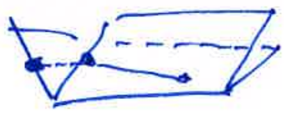
	Plan	
1	Key Problems	7.2d, 7.3b, 7.4, 8.1b, 8.2d, 8.3b, 9.2, 9.3a
2	Final exam	11/2018 3-4
3	Problems [WB]	7.9, 8.7, 8.12-13

1 Key problems

7.2 d) $f(x,y) = |x-y|$



Defn: f convex:

P, Q on the graph of f $\Rightarrow$ $[P,Q]$ lies over the graph (or on) (line segment)

Ok, f is convex

7.3 b) max/min $f = x^4 + y^4 + z^4$ when $2x^2 + y^2 + 2z^2 = 9$

i) D is bounded

$2x^2 \leq 9 : -\sqrt{9/2} \leq x \leq \sqrt{9/2}$

$y^2 \leq 9 : -3 \leq y \leq 3$

$2z^2 \leq 9 : -\sqrt{9/2} \leq z \leq \sqrt{9/2}$

EVT $\Rightarrow$ there is a max/min

ii) Candidate pts: $\mathcal{L} = x^4 + y^4 + z^4 - \lambda(2x^2 + y^2 + 2z^2)$

FOC
$$\begin{aligned}
\mathcal{L}'_x &= 4x^3 - 4\lambda x = \boxed{4x(x^2-\lambda) = 0} \\
\mathcal{L}'_y &= 4y^3 - 2\lambda y = \boxed{2y(2y^2-\lambda) = 0} \\
\mathcal{L}'_z &= 4z^3 - 4\lambda z = \boxed{4z(z^2-\lambda) = 0} \\
C &: \boxed{2x^2 + y^2 + 2z^2 = 9}
\end{aligned}$$

$x = 0$ or $x^2 = \lambda$

$y = 0$ or $y^2 = \lambda/2$

$z = 0$ or $z^2 = \lambda$

a) $x=y=z=0$: $\quad C \Rightarrow$ not possible

b) $x \neq 0,\ y=z=0$: $\quad x^2 = 9/2 = \lambda \Rightarrow (\pm\sqrt{9/2}, 0, 0;\ 9/2)$ $\quad f = \underline{\frac{81}{4}}$

$y \neq 0;\ x=z=0$: $\quad y^2 = 9,\ \lambda = 18$ $\quad (0, \pm 3, 0;\ 18)$ $\quad f = \underline{81}$

$z \neq 0,\ x=y=0$: $\quad z^2 = 9/2 = \lambda$ $\quad (0, 0, \pm\sqrt{9/2};\ 9/2)$ $\quad f = \underline{\frac{81}{4}}$

c) $x=0,\ y,z \neq 0$: $\quad \lambda = 2y^2 = z^2$ $\quad (0, \pm\sqrt{9/5}, \pm\sqrt{18/5};\ 18/5)$

C: $\quad y^2 + 2(2y^2) = 9$ $\quad f = \frac{81}{25} + \frac{18^2}{25}$

$5y^2 = 9$ $\quad = \frac{9^2 + 4 \cdot 9^2}{25}$

$y^2 = 9/5 \quad z^2 = 18/5$

$= 9^2/5 = \underline{81/5}$

$y = 0,\ x,z \neq 0$: $\quad x^2 = z^2 = \lambda$

C: $\quad 2x^2 + 0 + 2x^2 = 9$ $\quad (\pm 3/2, 0, \pm 3/2, 9/4)$

$x^2 = 9/4 = z^2$ $\quad f = (9/4)^2 \cdot 2$

$= \frac{81 \cdot 2}{16} = \underline{81/8}$

$\underline{z=0,\ x,y \neq 0}$: $\quad \Rightarrow (\pm\sqrt{18/5}, \pm\sqrt{9/5}, 0;\ 18/5)$

$f = \underline{81/5}$

d) $\underline{x,y,z \neq 0}$: $\quad \lambda = x^2 = z^2 = 2y^2$

C: $\quad 2 \cdot (2y^2) + y^2 + 2 \cdot (2y^2) = 9$

$9y^2 = 9$

$y^2 = 1 \quad x^2 = z^2 = 2 = \lambda$

$(\pm\sqrt{2}, \pm 1, \pm\sqrt{2};\ 2)$ $\quad f = 4 + 1 + 4 = \underline{9}$

$\underline{\text{Concl}}$: $\quad f_{max} = \underline{\underline{81}}$

$f_{min} = \underline{\underline{9}}$

7.4 b) max $f = \ln(xyz)$ when $2x^2 + y^2 + 2z^2 \le 6$

$xyz > 0$

KT-pb in std. form.

$\ln(u)$, $u = xyz$

} derivative

$\frac{1}{u} \cdot u'$

$$L = \ln(xyz) - \lambda(2x^2 + y^2 + 2z^2)$$

FOC

$$L'_x = \frac{1}{\cancel{x}yz} \cdot \cancel{yz} - 4\lambda x = \frac{1}{x} - 4\lambda x = 0$$

$$L'_y = \frac{1}{x\cancel{y}z} \cdot \cancel{xz} - 2\lambda y = \frac{1}{y} - 2\lambda y = 0$$

$$L'_z = \frac{1}{xy\cancel{z}} \cdot \cancel{xy} - 4\lambda z = \frac{1}{z} - 4\lambda z = 0$$

C:	$2x^2+y^2+2z^2=6$	$2x^2+y^2+2z^2<6$
CSC:	$\lambda \ge 0$	$\lambda = 0$

$\frac{1}{x} = \frac{1}{y} = \frac{1}{z} = 0$ not possible

$\frac{1}{x} - 4\lambda x = 0 \quad | \cdot x$

$1 - 4\lambda x^2 = 0$

$4\lambda x^2 = 1$

$x^2 = 1/4\lambda$

$y^2 = 1/2\lambda$

$z^2 = 1/4\lambda$

$2 \cdot \left(\frac{1}{4\lambda}\right) + \left(\frac{1}{2\lambda}\right) + 2 \cdot \left(\frac{1}{4\lambda}\right) = 6$

$\frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} = 6 \quad | \cdot 2\lambda$

$\frac{3}{12} = \frac{6 \cdot 2\lambda}{12}$

$\lambda = 1/4$

$\Rightarrow x^2 = 1$

$y^2 = 2$

$z^2 = 1$

$(x, y, z; \lambda) = (\pm 1, \pm\sqrt{2}, \pm 1; 1/4)$

$f = \ln(1 \cdot \sqrt{2} \cdot 1)$

$= \ln(\sqrt{2}) = \frac{1}{2}\ln 2$

at $(1, \sqrt{2}, 1)$, $(1, -\sqrt{2}, -1)$,

$(-1, \sqrt{2}, -1)$, $(-1, -\sqrt{2}, 1)$

EVT: i) $2x^2+y^2+2z^2 \le 6 \Rightarrow$ bounded set,

$-\sqrt{3} \le x \le \sqrt{3}$

$-\sqrt{6} \le y \le \sqrt{6}$

$-\sqrt{3} \le z \le \sqrt{3}$

ii) $xyz > 0 \Rightarrow$ not closed

close to $xyz=0$: $f=\ln(xyz) \to -\infty$

Conclusion: Even though D is not closed, this ensures that there is a max

$\Downarrow$

$f_{max} = \ln(\sqrt{2}) = \frac{1}{2}\ln 2$

8.1 b) Check if (2,0,0) is min for

min $f = x^2+y^2+z^2$ when $3x^2+2y^2+2z^2 \ge 12$

$\Updownarrow$

Check if $\underline{(2,0,0)}$ is max for:

max $-f = -x^2-y^2-z^2$ when $-3x^2-2y^2-2z^2 \le -12$

SOC: $\boxed{h(x,y,z) = L(x,y,z;\lambda^*) \text{ concave?}}$

$$L = -x^2-y^2-z^2-\lambda(-3x^2-2y^2-2z^2)$$

$$L'_x = -2x + \lambda \cdot 6x = 0$$

$\underline{x=2}$: $-4+12\lambda = 0$

$12\lambda = 4$

$\lambda = \underline{1/3}$

$\Downarrow$

$$h = \cancel{-x^2} - y^2 - z^2 + \cancel{x^2} + \frac{2}{3}y^2 + \frac{2}{3}z^2$$

$$= -\frac{1}{3}y^2 - \frac{1}{3}z^2$$

$$H(h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix}$$

$\lambda_1 = 0 \le 0$

$\lambda_2 = -2/3 \le 0$

$\lambda_3 = -2/3 \le 0$

h is concave

$\Downarrow$ SOC

(2,0,0) is max for $-f$

$\Updownarrow$

$\underline{(2,0,0) \text{ is min for } f}$

8.2 d) Find adm. pts that does not satisfy NDCQ:

$g_1 \to xy - zw = 1$
$g_2 \to x+y+z+w = 4$

$$J = \begin{pmatrix} y & x & -w & -z \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

NDCQ: $\operatorname{rk} J = 2$

NDCQ fails: $\operatorname{rk} J < 2 \iff$ all 2-minors are zero

$M_{12,12} = y - x = 0$ $\quad y = x$

$M_{12,23} = x + w = 0$ $\quad w = -x$

$M_{12,34} = -w + z = 0$ $\quad w = z = -x \Rightarrow z = -x$

$$J = \begin{pmatrix} x & x & x & x \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

rk 1

$(x, x, -x, -x)$

C2: $x + x - x - x = 4$

$0 \cdot x = 4$

impossible

Concl: No adm. pts with $\operatorname{rk} J < 2$.

83 b) max $f = xz + yw$ wh $\begin{cases} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{cases}$

bounded

$-1 \le x, y \le 1$

$-3 \le z \le 3$

$-2 \le w \le 2$

$\Downarrow$ EVT

there is a max

KT, std. form

$$\mathcal{L} = xz + yw - \lambda_1(x^2+y^2) - \lambda_2(4z^2+9w^2)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= z - \lambda_1 \cdot 2x = 0 \\ \mathcal{L}'_y &= w - \lambda_1 \cdot 2y = 0 \\ \mathcal{L}'_z &= x - \lambda_2 \cdot 8z = 0 \\ \mathcal{L}'_w &= y - \lambda_2 \cdot 18w = 0 \end{aligned}}$$

FOC

(1) $z = 2\lambda_1 \cdot x$

(3) $x - 8\lambda_2 \cdot (2\lambda_1 x) = 0$

$x \cdot (1 - 16\lambda_1\lambda_2) = 0$

a) $\boxed{x = 0, \ z = 0}$ or b) $\boxed{\lambda_1\lambda_2 = 1/16, \ z = 2\lambda_1 \cdot x}$

(2) $w = 2\lambda_1 \cdot y$

(4) $y - 18\lambda_2(2\lambda_1 y) = 0$

$y(1 - 36\lambda_1\lambda_2) = 0$

c) $\boxed{y = 0, \ w = 0}$ or d) $\boxed{\lambda_1\lambda_2 = 1/36, \ w = 2\lambda_1 y}$

a) + c): $(x,y,z,w) = (0,0,0,0)$ $\quad \lambda_1 = 0, \ \lambda_2 = 0$ $\quad f = 0$

both C ok and non-binding

a) + d): $x = z = 0$, $\lambda_1\lambda_2 = 1/36$, $w = 2\lambda_1 \cdot y$

CSC: $\lambda_1, \lambda_2 > 0$

C: $x^2+y^2 = 1$ $\quad y = \pm 1$

$4z^2+9w^2 = 36$ $\quad w = \pm 2$

$(0, 1, 0, 2; 1, 1/36)$ $\quad f = 2$

$(0, -1, 0, -2; 1, 1/36)$ $\quad f = 2$

b) + c): $y = w = 0$, $\lambda_1 \lambda_2 = 1/16$, $z = 2\lambda_1 x$

CSC: $\lambda_1, \lambda_2 > 0$

C: $x^2 + y^2 = 1$, $4z^2 + 9w^2 = 36$ $\quad x = \pm 1$, $z = \pm 3$

$(1, 0, 3, 0 ; 3/2, 1/24)$ $f = 3$

$(-1, 0, -3, 0 ; 3/2, 1/24)$ $f = 3$

b) + d) $\lambda_1 \lambda_2 = 1/36$, $\lambda_1 \lambda_2 = 1/16$ impossible

$\Downarrow$

Best candidate pts: $(1, 0, 3, 0 ; 3/2, 1/24)$ $f = 3$

$(-1, 0, -3, 0 ; 3/2, 1/24)$ $f = 3$

Alt I: EVT $\Rightarrow$ there is a max

Must check NDCQ. ✓ $\Rightarrow$ $f_{max} = 3$

Alt 2: SOC

Is $h = xz + yw - \frac{3}{2}(x^2 + y^2) - \frac{1}{24}(4z^2 - 9w^2)$ concave?

NDCQ: $x^2 + y^2 = 1$, $4z^2 + 9w^2 = 36$ $\quad$ $\text{rk}\begin{pmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 8z & 18w \end{pmatrix} = 2$

A) fails if $\text{rk} J < 2$: $16xz = 0$, $36xw = 0$, $16yz = 0$, $36yw = 0$

$(x = 0 \text{ or } z = w = 0)$ $(y = 0 \text{ or } z = w = 0)$

$x = 0, y = \pm 1$ $\quad z = w = 0$ not possible

B) $x^2 + y^2 = 1$, $\dots < 36$ not possible

$\text{rk}(2x \;\; 2y \;\; 0 \;\; 0) = 1$

C) $\dots < 1$, $4z^2 + 9w^2 = 36$ $\quad \text{rk}(0 \;\; 0 \;\; 8z \;\; 18w) = 1$ not possible

not possible

9.2 max $f = 4x^3 - 2y^3 + z^3$ when $x^3 + y^3 + z^3 \le 8$

a) $L = 4x^3 - 2y^3 + z^3 - \lambda(x^3 + y^3 + z^3)$

FOC: $L'_x = 12x^2 - 3\lambda x^2 = 0$ $\quad 3x^2(4-\lambda) = 0$

$L'_y = -6y^2 - 3\lambda y^2 = 0$ $\quad 3y^2(-2-\lambda) = 0$

$L'_z = 3z^2 - 3\lambda z^2 = 0$ $\quad 3z^2(1-\lambda) = 0$

$x = 0$ or $\lambda = 4$

$y = 0$ ~~or $\lambda = -2$~~

$z = 0$ or $\lambda = 1$

a) $x = 0, y = 0, z = 0$

C: $0 < 8$ $\quad \lambda = 0$

$(0,0,0;0)$ $\quad f = 0$

b) $x = 0, y = 0, z \neq 0$:

$\lambda = 1$, $\quad z^3 = 8$ $\Rightarrow (0,0,2;1)$ $\quad f = 8$

$z = 2$

c) $x \neq 0, y = 0$: $\quad$ C: $x^3 = 8$

$\lambda = 4, z = 0$ $\quad x = 2$

$(2,0,0;4)$ $\quad f = 32$

Best candidate pt.

b) Not max: Possibilities: i) ~~there is an adm pt. where NDCQ fails with $f > 32$~~

NDCQ fails:

$x^3 + y^3 + z^3 \le 8$ $\quad x^3+y^3+z^3 = 8$: rk $(3x^2 \; 3y^2 \; 3z^2) = 1$ ok since $(x,y,z) \neq (0,0,0)$

$x^3+y^3+z^3 < 8$: ok

ii) there is no max

$x^3+y^3+z^3 \leq 8$: not bounded $(x,y,z \to -\infty)$

$f = 4x^3 - 2y^3 + z^3$: $(x,0,0)$ $f = 4x^3 \to -\infty$
$x \to -\infty$

$(0,y,0)$ $f = -2y^3 \to \infty$
$y \to -\infty$

There is no max: $f(0,-10,0) = 2000$

Necessary condition:

$\underline{x}^*$ is max $\Longrightarrow$ FOC + C + CSC must holds

9.3. max $f = 2x^2 - 4y^2 - 2z^2$ when $x^4 + y^4 + z^4 \le 16$

KT prob. std. form

a)

$$\mathcal{L} = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4)$$

FOC:

$$L'_x = 4x - \lambda \cdot 4x^3 = \boxed{4x(1 - \lambda x^2) = 0}$$
$$L'_y = -8y - \lambda\, 4y^3 = \boxed{4y(-2 - \lambda y^2) = 0}$$
$$L'_z = -4z - \lambda \cdot 4z^3 = \boxed{4z(-1 - \lambda z^2) = 0}$$

C: $\boxed{x^4 + y^4 + z^4 \le 16}$

CSC: $\boxed{\lambda \ge 0,\quad \lambda = 0 \text{ if } x^4 + y^4 + z^4 < 16}$

Solve FOC + C + CSC:

FOC: $\cancel{x = 0}$ or $\lambda = 1/x^2$
$y = 0$ or $\cancel{\lambda = -2/y^2}$
$z = 0$ or $\cancel{\lambda = -1/z^2}$
$\Rightarrow y = z = 0$

a) $x = 0$: $x = y = z = 0$
C: $0 < 16$ ok, $\lambda = 0$ $\Big\}$ $(0,0,0;0)$ $f = 0$

b) $x \ne 0$: $\lambda = 1/x^2 > 0$ $\quad x^4 + y^4 + z^4 = 16$
$y = z = 0$
$x^4 = 16$
$x^2 = \pm\sqrt{16} = \pm 4 = 4$
$x = \pm 2$ $\quad \lambda = 1/4$

$(\pm 2, 0, 0; 1/4)$ $f = 8$

Check that $f(\pm 2,0,0)=8$ is max:

C: $\boxed{x^4+y^4+z^4 \leq 16}$ Bounded: $x^4 \leq 16$ $x^2 \leq 4$

$$\boxed{\begin{array}{c} -2 \leq x \leq 2 \\ -2 \leq y \leq 2 \\ -2 \leq z \leq 2 \end{array}}$$

EVT $\Leftarrow$ there is a max

Cand. pts:

i) Ord. cand pts: $f(0,0,0)=0$, $f(\pm 2,0,0)=8$

ii) Adm. pts where NDCQ fails: No pts

$x^4+y^4+z^4=16$: NDCQ: $rk\begin{pmatrix}4x^3 & 4y^3 & 4z^3\end{pmatrix}=1$ only fails for $(0,0,0)$,

$x^4+y^4+z^4<16$: No NDCQ. not binding

Concl: $f_{max}=8$ max pt: $(\pm 2,0,0)$