	Plan	
1	Key Problems	6.3a, 6.4be
2	Midterm	10/2018 Q8, 01/2018 Q5
3	Problems [WB]	4.3b, 4.11, 5.7, 5.13, 5.14, 6.4, 6.5, 6.9, 6.21, 6.22, 6.24, 7.1, 7.9, 7.10

① Key problems

6.3a) $f = xy + xz - yz$

$f'_x = y + z = 0$

$f'_y = x - z = 0$

$f'_z = x - y = 0$

$\Downarrow$

$(x,y,z) = (0,0,0)$

$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} = -1\cdot 1 + 1(-1) \neq 0$

$A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$H(f) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ $D_1 = 0$, $D_2 = -1$, Ind. $\Rightarrow$ $(0,0,0)$ saddle pt.

no global max/min

6.4 b) $f(x,y,z) = e^{x-2y+z} = e^u$, $u = x - 2y + z$

$f'_x = e^u \cdot 1$

$f'_y = e^u \cdot (-2)$

$f'_z = e^u \cdot 1$

$H(f) = \begin{pmatrix} e^u \cdot 1 & e^u \cdot (-2) & e^u \cdot 1 \\ e^u \cdot (-2) & e^u (-2)^2 & e^u \cdot (-2) \cdot 1 \\ e^u \cdot 1 & e^u (-2) \cdot 1 & e^u \cdot 1 \end{pmatrix}$

$= e^u \cdot \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

$D_1 = 1 \cdot e^u > 0$

$D_2 = 0 \cdot (e^u)^2 = 0$

$D_3 = (e^u)^3 \cdot [1 \cdot 0 + 2 \cdot 0 + 1 \cdot 0] = 0$

rk $H(f) = 1$ $\xRightarrow{\text{Rec}}$ $H(f)$ pos. semidefin. $\Rightarrow$ f convex

Note: f convex and concave $\Leftrightarrow$ f is linear

$H(f) = 0$

e) $f(x,y,z) = \dfrac{xy + xz + yz}{xyz}$ $\quad (x,y,z > 0)$

$= \frac{1}{z} + \frac{1}{y} + \frac{1}{x}$

$= x^{-1} + y^{-1} + z^{-1}$

$f'_x = -1 \cdot x^{-2}$

$f'_y = -1 \cdot y^{-2}$

$f'_z = -1 \cdot z^{-2}$

$$H(f) = \begin{pmatrix} 2x^{-3} & 0 & 0 \\ 0 & 2y^{-3} & 0 \\ 0 & 0 & 2z^{-3} \end{pmatrix}$$

$D_1 = 2/x^3 > 0$ $\quad$ f convex

$D_2 = 4/x^3y^3 > 0$

$D_3 = 8/x^3y^3z^3 > 0$

Note: convex/concave functions can only be defined over convex sets

② Midterm exams

10/2018, Q8:

$$f(x,y,z) = 1 - (x-y+z)^4 = \underline{1-u^4, \quad u = x-y+z}$$

$x-y+z=0$: $f=1 \leftarrow$ global max. value

$x-y+z\neq 0$: $f<1$

$(1,1,0)$: $x-y+z=0$ $\boxed{\text{one of the global max pt.}}$

$$f'_x = -4u^3\cdot 1 = \underline{-4u^3}$$
$$f'_y = -4u^3\cdot(-1) = \underline{4u^3}$$
$$f'_z = -4u^3\cdot 1 = \underline{-4u^3}$$

$$H(f) = \begin{pmatrix} -12u^2\cdot 1 & -12u^2\cdot(-1) & -12u^2\cdot 1 \\ 12u^2\cdot 1 & 12u^2\cdot(-1) & 12u^2\cdot 1 \\ -12u^2\cdot 1 & -12u^2\cdot(-1) & -12u^2\cdot 1 \end{pmatrix} = 12u^2\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$D_1 = \underline{12u^2\cdot(-1) \leq 0}$$
$$D_2 = (12u^2)^2\cdot 0 = 0$$
$$D_3 = (12u^2)^3\cdot 0$$

$u=0$: $H(f) = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$ pos. + neg. semidef.

$u\neq 0$: $\operatorname{rk} H(f) = 1$

$\Downarrow$ RRC, $D_1<0$

$H(f)$ ~~neg.~~ semidef.

$H(f)$ neg. semidefin for all (x,y,z)

$\Downarrow$

f concave (D)

01/2018, Q5:

$$A = \begin{pmatrix} 1 & s & -s^2 \\ 0 & 0 & s \\ 0 & 0 & 1 \end{pmatrix}$$

$s=0$: A diagonalizable since it is symm.

Note: $\lambda=1, \lambda=0, \lambda=1$ are the eigenvalues since A is upper triangular

$$\begin{vmatrix} 1-\lambda & s & -s^2 \\ 0 & -\lambda & s \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\cdot(-\lambda)(1-\lambda) = 0$$

$\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 1$

$\lambda = 1$: ($m=2$)

$\lambda = 0$: ($m=1$)

$$\begin{pmatrix} 0 & s & -s^2 \\ 0 & -1 & s \\ 0 & 0 & 0 \end{pmatrix}$$

2 free variables for all s

$\Uparrow$

rk = 1 for all s

$\Uparrow$

$$\begin{vmatrix} s & -s^2 \\ -1 & s \end{vmatrix} = s^2 - s^2 = 0 \Leftrightarrow$$ all 2-minors are zero for all s

$$\begin{pmatrix} 0 & -1 & s \\ 0 & s & -s^2 \\ 0 & 0 & 0 \end{pmatrix} \left.\right] s$$

$\downarrow$

$$\begin{pmatrix} 0 & -1 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ rk = 1, x_1, x_3 free (for all s)

echelon form

Conclusion: A diag. for all s (A)

③ Problems [WB]

4.3 b) $A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix}$

$$\begin{vmatrix} 4-\lambda & 1 & 2 \\ 0 & 3-\lambda & 0 \\ 1 & 1 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\cdot(\lambda^2 - 9\lambda + 18) = 0$$

$\lambda_1 = 3$, $\lambda_2 = 3$, $\lambda_3 = 6$

$\lambda = 3$: $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ \cancel{1} & \cancel{1} & \cancel{2} \end{pmatrix}$ $1\cdot x + 1\cdot y + 2z = 0$

$\Rightarrow x = -y - 2z$, y, z free

$$\underline{v} = \begin{pmatrix} -y-2z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 6$: $\begin{pmatrix} -2 & 1 & 2 \\ 0 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 0 \\ -2 & 1 & 2 \end{pmatrix}$ 2

$\to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 0 \\ \cancel{0} & \cancel{3} & \cancel{0} \end{pmatrix}$ $x = -y + z = z$

$-3y = 0 \Rightarrow y = 0$

z free

$$\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

4.4

$$A = \begin{pmatrix} 1 & 1 & -4 \\ 0 & t+2 & t-8 \\ 0 & -5 & 5 \end{pmatrix}$$

i) $|A| = 1 \cdot ((t+2)\cdot 5 - (t-8)(-5))$

$= 5t+10 + 5t - 40 = \underline{10t-30} = 10(t-3)$

$\underline{t \neq 3}$: $|A| \neq 0 \Rightarrow \underline{\text{rk } A = 3}$

$\underline{t=3}$: $\begin{pmatrix} 1 & 1 & -4 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{pmatrix}$ $\underline{\text{rk } A = 2}$

ii) $\begin{vmatrix} 1-\lambda & 1 & -4 \\ 0 & \boxed{t+2-\lambda} & \boxed{t-8} \\ 0 & \boxed{-5} & \boxed{5-\lambda} \end{vmatrix} = 0$

$(1-\lambda)\cdot(\lambda^2 - (t+7)\lambda + 10(t-3)) = 0$

$\underline{\lambda = 1}$ or $\lambda = \dfrac{t+7 \pm \sqrt{(t+7)^2 - 4\cdot 1\cdot 10(t-3)}}{2}$

$= \dfrac{t+7 \pm \sqrt{t^2+14t+49-40t+120}}{2}$

$= \dfrac{t+7 \pm \sqrt{t^2-26t+169}}{2} = \dfrac{t+7 \pm \sqrt{(t-13)^2}}{2}$

$= \dfrac{(t+7) \pm (t-13)}{2} = \underline{t-3}, \underline{10}$

iii) $\underline{\text{When is } A \text{ diag:}}$

$\lambda = 1, t-3, 10$

$\underline{t \neq 4, 13}$: all eigval. have $m=1$ $\Rightarrow$ $\underline{A \text{ diag.}}$

$t = 4, 13$:

$\underline{t=4}$: $\lambda = \underbrace{1, 1}_{m=2}, 10$

$\underline{t=4, \lambda=1}$: $\begin{pmatrix} 0 & 1 & -4 \\ 0 & 5 & -4 \\ 0 & -5 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -4 \\ 0 & 0 & 16 \\ 0 & 0 & -16 \end{pmatrix}$

$\left.\begin{array}{c}\text{one free var} \\ m=2\end{array}\right\}$ not dig for $\underline{t=4}$

$\underline{t=13}$: $\lambda = 1, \underbrace{10, 10}_{m=2}$

$\underline{t=13, \lambda=10}$: $\begin{pmatrix} -9 & 1 & -4 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{pmatrix}$ one free var, $m=2$ $\Downarrow$ $\underline{\text{not diag.}}$

$\underline{5.7}$. $\underline{A^T A \text{ pos defn. for any invertible matrix } A}$

A
$n \times n$
$|A| \neq 0$

① $\underline{A^T A \text{ pos. semidef.}}$: ✓

$\underline{x}^T (A^T A) \underline{x} = (A\underline{x})^T \cdot (A\underline{x})$
$= \underline{h}^T \cdot \underline{h} = (h_1 \dots h_n) \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$
$= h_1^2 + h_2^2 + \dots \geq 0$

② $\underline{A^T A \text{ pos. defn}}$:

$\lambda_1, \lambda_2, \dots, \lambda_n$ eigval. of $A^T A$
$\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$ from ①

$|A^T A| = |A^T| \cdot |A| = |A| \cdot |A| = |A|^2 \neq 0$

$\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n \neq 0 \Rightarrow \lambda_1, \lambda_2, \dots, \lambda_n > 0 \Rightarrow$ $A^T A$ pos defn.

$A^T A$ pos. defn.

5.13

$$A = \begin{pmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{pmatrix}$$

$t = 8 - \lambda$

a) $|A| = t \cdot (t^2 - 1) - 1 \cdot (t-1) + 1 \cdot (1-t)$

$= t(t^2-1) - t + 1 - t + 1$

$= t(t-1)(t+1) - 2(t-1)$

$= (t-1)(t(t+1) - 2) = (t-1)(t^2 + t - 2)$

$= (t-1)(t+2)(t-1)$

$= (t-1)^2 \cdot (t+2)$

rk $(A - \lambda I)$ when $\lambda = t-1$

$$= \text{rk} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \underline{\underline{1}}$$

$t = 8 - \lambda$

b) $t = 8$: $A = \begin{pmatrix} 8 & 1 & 1 \\ 1 & 8 & 1 \\ 1 & 1 & 8 \end{pmatrix}$ symm $\Downarrow$ diag

$$\begin{vmatrix} 8-\lambda & 1 & 1 \\ 1 & 8-\lambda & 1 \\ 1 & 1 & 8-\lambda \end{vmatrix} = 0$$

$(8-\lambda-1)^2 \cdot (8-\lambda+2) = 0$

$(7-\lambda)^2 \cdot (10-\lambda) = 0$

$\lambda_1 = \lambda_2 = 7, \; \lambda_3 = 10$

$\lambda = 7$ has to be an eigenvalue ($m=2$) since

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$7 + 7 + \lambda_3 = 24$

$\lambda_3 = 10$

5.14

$$T = \frac{1}{t+n-1}\begin{pmatrix} t & 1 & \cdots & 1 \\ 1 & t & & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & t \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{pmatrix} = \frac{1}{n}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\frac{1}{t+n-1}\begin{pmatrix} t & 1 & \cdots & 1 \\ 1 & t & & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{1}{t+n-1}\begin{pmatrix} t+n-1 \\ t+n-1 \\ \vdots \\ t+n-1 \end{pmatrix}$$

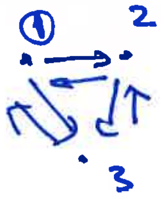
Concl: $\underline{v} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector of T $= \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$\Updownarrow$

multiple of eq. state

$$\underline{v} = \frac{1}{n}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{eq. state}$$

$t=0$:



is regular

6.4 iii) $f(x,y) = \ln(x^2+y^2+1)$

$= \ln(u)$, $u = x^2+y^2+1$

$$f'_x = \frac{1}{u}\cdot 2x = \frac{2x}{u} = \frac{2x}{x^2+y^2+1}$$

$$f'_y = \frac{1}{u}\cdot 2y = \frac{2y}{u} = \frac{2y}{x^2+y^2+1}$$

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix}$$

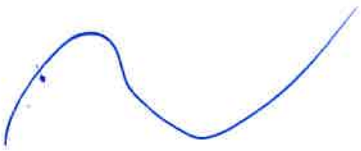
$$f''_{xx} = \frac{2\cdot u - 2x\cdot 2x}{u^2} = \frac{2(x^2+y^2+1)-4x^2}{u^2}$$

$$= \frac{-2x^2+2y^2+1}{u^2} = D_1$$

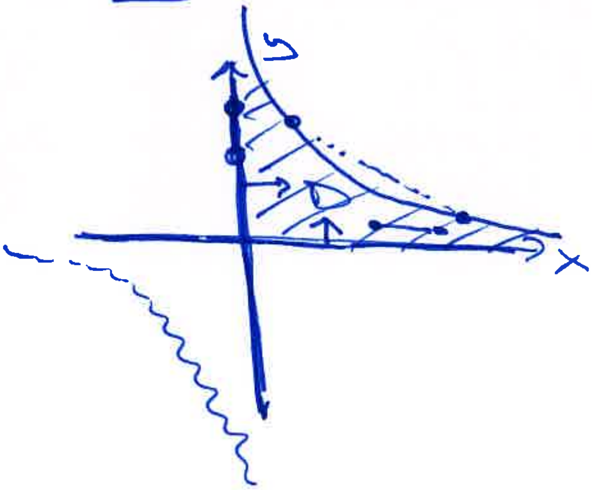
f convex $\Rightarrow D_1 \geq 0$ for all x,y : $D_1(1,0) = \frac{-2+0+1}{2^2} < 0$

f concave $\Rightarrow D_1 \leq 0$ for all x,y $D_1(0,1) = \frac{2\cdot 1+1}{2^2} > 0$

f is not convex, not concave



6.5. $D = \{ (x,y) : x \geq 0,\ y \geq 0,\ xy \leq 1 \}$



$x \geq 0$: $x = 0$

$y \geq 0$: $y = 0$

$xy \leq 1$: $xy = 1$

$y = 1/x$

$xy < 1$ $\quad 1 \cdot \frac{1}{x}$

$y < 1/x$

Is D convex?

P, Q in D

$\Downarrow$

[P, Q] in D

No!

D is convex

$\Updownarrow$

$y = 1/x$ is concave fn.

$y' = -1/x^2 \quad y'' = 2/x^3 > 0$

$H = (2/x^3)$ pos. defin

$y = 1/x$ convex

Reminder:

f convex $\Leftrightarrow$ H(x) pos. semidef. for all x

f concave

P, Q on f $\Rightarrow$ [P, Q] under graph

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