	Plan	
1	Key Problems	1.3, 3.3c, 3.4b, 4.1cdf, 4.2de, 4.3, 4.4
2	Problems [E]	1.9, 1.14, 2.4, 2.5, 2.6, 2.11, 2.14, 3.10, 3.11, 3.12b, 3.13, 3.14, 3.15
3	Problems [WB]	4.4, 4.11, 4.12

1 Key problems:

1.3. $A = \begin{pmatrix} 1 & 1 & 4 & -1 \\ t & 5 & -1 & 4 \\ 7 & 6 & 3 & 3 \end{pmatrix} \begin{matrix} \\ ]-t \\ ]-7 \end{matrix}$

$$\to \begin{pmatrix} 1 & 1 & 4 & -1 \\ 0 & 5-t & -1-4t & 4+t \\ 0 & -1 & -25 & 10 \end{pmatrix} \updownarrow \to \begin{pmatrix} 1 & 1 & 4 & -1 \\ 0 & -1 & -25 & 10 \\ 0 & 5-t & -1-4t & 4+t \end{pmatrix}](5-t)$$

$$\to \begin{pmatrix} 1 & 1 & 4 & -1 \\ 0 & -1 & -25 & 10 \\ 0 & 0 & 21(t-6) & -9(t-6) \end{pmatrix}$$

$$\left\{ \begin{aligned} &-1-4t-25\cdot(5-t) \\ &= 21t-126 = 21(t-6) \\ &4+t+10(5-t) \\ &= -9t+54 = -9(t-6) \end{aligned} \right.$$

$t=5$: $(t \neq 6)$ $\left(\begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right)$ one free var, solutions = a line in $\mathbb{R}^4$

$t=6$: two free vars, solutions = a plane in $\mathbb{R}^4$

$t \neq 6$: one free var, solutions = a line in $\mathbb{R}^4$

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3.3 c

$$A = \begin{pmatrix} 1 & a & b \\ a & b & 1 \end{pmatrix} \qquad \text{rk } A = ?$$

$\text{rk } A = 2 \iff$ there is a non-zero 2-minor

$\boxed{\text{rk } A < 2 \iff \text{all 2-minors are zero}}$

$$\left.\begin{aligned} M_{12,12} &= b - a^2 = 0 \\ M_{12,23} &= a - b^2 = 0 \\ M_{12,13} &= 1 - ab = 0 \end{aligned}\right\} \qquad \left.\begin{aligned} &b = a^2 \\ &a = (a^2)^2 \\ &a = a^4 \\ &a - a^4 = 0 \\ &a(1-a^3) = 0 \\ &\cancel{a = 0}, \quad a^3 = 1 \\ &\cancel{b = 0}, \quad a = 1 \\ &\qquad\quad\;\; b = 1 \end{aligned}\right\} \quad \text{rk } A < 2$$

Conclusion: $\text{rk } A = \begin{cases} 2, & (a,b) \neq (1,1) \\ 1, & a = b = 1 \end{cases}$

3.4 b)

$$\left.\begin{aligned} x + 4y + 5z - 3w &= 6 \\ 2x + 7y + z &= 4 \\ x + 5y + 4z - 8w &= 1 \end{aligned}\right\} \iff \begin{array}{cccc|c} x & y & z & w & \\ \hline 1 & 4 & 5 & -3 & 6 \\ 2 & 7 & 1 & 0 & 4 \\ 1 & 5 & 4 & -8 & 1 \end{array}$$

$$M_{123,123} = \begin{vmatrix} 1 & 4 & 5 \\ 2 & 7 & 1 \\ 1 & 5 & 4 \end{vmatrix} = 1 \cdot 23 - 4 \cdot 7 + 5 \cdot 3 = 10 \neq 0$$

Infinitely many solutions, one free variable (w)

$M_{123,124} \neq 0 \Rightarrow z$ can be free

$M_{123,134} \neq 0 \Rightarrow y$ can be free

$M_{123,234} \neq 0 \Rightarrow x$ can be free

4.1

c) $A = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} 2-\lambda & -4 \\ 3 & -1-\lambda \end{vmatrix} = 0$

$$\lambda^2 - \lambda + 10 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-40}}{2}$$

no real solutions

$\Downarrow$

no eigenvalues, no eigenvectors

d) $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 5-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} = 0$

$$(5-\lambda) \cdot \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(\lambda^2 - 8\lambda + 15) = 0$$

$\lambda = 5$, $\lambda = 5$, $\lambda = 3$ $\Rightarrow$ $\lambda_1 = \lambda_2 = 5$ (mult=2), $\lambda_3 = 3$ (mult=1)

E_5: $\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{1} & \cancel{0} & \cancel{-1} & \cancel{0} \end{array}\right)$ $\quad$ $x = z$, $y = y$, $z = z$ $\quad$ $\underline{v} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$0 \cdot y = 0$ $\quad$ $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix}$

Basis of E_5: $\left\{ \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

E_3: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} \end{array}\right)$ $\quad$ $x + z = 0$, $2y = 0$ $\quad$ $x = -z$, $y = 0$, $z = z$ $\quad$ $\underline{v} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Basis of E_3: $\left\{ \underline{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

f) $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(2-\lambda)(2-\lambda) = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = 2$ (mult = 3)

E_2: $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ x free, $y+z=0$, $\cancel{z}=0$ $x=x$, $y=0$, $z=0$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Base of E_2: $\left\{ \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

4.2. d) Yes, $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$P = (\underline{v}_1 | \underline{v}_2 | \underline{v}_3) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

f) No, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$P = \begin{pmatrix} 1 & ? & ? \\ 0 & & \\ 0 & & \end{pmatrix}$

e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ Yes

$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Alternative method to find eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \quad \leftarrow$$

$\lambda = 1$: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ det $= 0$

rk $= 1$, two free var.

$\Downarrow$

$\lambda = 1$ eigenvalue , $1 \leq \dim E_1 \leq m$

$\dim E_1 = \#$ free vars $= 2$, $m = \text{mult}(\lambda)$

$\underbrace{}_{m(\lambda=1) \geq 2}$

$\lambda_1 = \lambda_2 = 1$, $\lambda_3 = ?$

$\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(A) = 6$

$1 + 1 + \lambda_3 = 6$

$\underline{\lambda_3 = 4}$

4.3

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rk} A = 2$

$A \cdot \underline{v} = 0$ has two free var's.

$\lambda = 0$ is an eigenvalue, and $\text{mult}(\lambda = 0) \geq 2$

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 0 & 0 & -1 \\ 0 & 1-\lambda & -1 & 0 \\ 0 & -1 & 1-\lambda & 0 \\ -1 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$- \begin{vmatrix} -1 & 0 & 0 & 1-\lambda \\ 0 & 1-\lambda & -1 & 0 \\ 0 & -1 & 1-\lambda & 0 \\ 1-\lambda & 0 & 0 & -1 \end{vmatrix} \; (1-\lambda) = 0$$

$$- \begin{vmatrix} -1 & 0 & 0 & 1-\lambda \\ 0 & 1-\lambda & -1 & 0 \\ 0 & -1 & 1-\lambda & 0 \\ 0 & 0 & 0 & (1-\lambda)^2-1 \end{vmatrix} = 0$$

$$\to -(-1) \cdot \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & (1-\lambda)^2-1 \end{vmatrix} = 0$$

$$[(1-\lambda)^2-1] \cdot [(1-\lambda)^2-1] = 0$$

$$(\lambda^2-2\lambda)(\lambda^2-2\lambda) = 0$$

$$\lambda(\lambda-2) \cdot \lambda(\lambda-2) = 0$$

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = 2$$

A symmetric $\Rightarrow$ A diagonalizable

$\to \lambda_1 = \lambda_2 = 0$

$\lambda_3 = ? \quad \lambda_4 = ?$

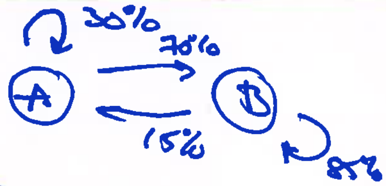
$$\underset{\overset{\shortparallel}{0}}{\lambda_1} + \underset{\overset{\shortparallel}{0}}{\lambda_2} + \lambda_3 + \lambda_4 = tr(A) = 4$$

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 = |A| = 0$$

$$\underline{\lambda_3 + \lambda_4 = 4}$$

$$0 = |A| = |A - \underset{\lambda = 0}{\lambda} I|$$

4. a) $A = \begin{pmatrix} 0.30 & 0.15 \\ 0.70 & 0.85 \end{pmatrix}$



$u_0 \to A\cdot u_0 \to A^2\cdot u_0 \to \dots$

$\dots \to A^m \cdot u_0$

Eigenvalues:

$$|A-\lambda I| = \begin{vmatrix} 0.3-\lambda & 0.15 \\ 0.70 & 0.85-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1.15\lambda + 0.15 = 0 \qquad 0.3\cdot 0.85 - 0.15\cdot 0.7$$

$\lambda = 1$, $\lambda = 0.15$

Eigenvectors:

E_1: $\left(\begin{array}{cc|c} -0.7 & 0.15 & 0 \\ \cancel{0.7} & \cancel{-0.15} & 0 \end{array}\right)$ $\quad -0.70x + 0.15y = 0 \Rightarrow x = \frac{0.15y}{0.70}$

$x = 15$, $y = 70$

$\Rightarrow \underline{u} = \begin{pmatrix} 15 \\ 70 \end{pmatrix}\cdot s \quad \Big| \quad \underline{u} = \begin{pmatrix} \frac{15}{70}y \\ y \end{pmatrix} = \begin{pmatrix} 15/70 \\ 1 \end{pmatrix}\cdot y$

$E_{0.15}$: $\left(\begin{array}{cc|c} 0.15 & 0.15 & 0 \\ \cancel{0.70} & \cancel{0.70} & 0 \end{array}\right)$

$0.15x + 0.15y = 0 \qquad x + y = 0 \qquad x = -y$

$$\underline{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Compute A^m: A is diagonalizable: $D = \begin{pmatrix} 1 & 0 \\ 0 & 0.15 \end{pmatrix}$ $\quad P = \begin{pmatrix} 15 & -1 \\ 70 & 1 \end{pmatrix}$

$\Updownarrow$

$P\cdot | \; P^{-1}AP = D \qquad AP = PD \; |\cdot P^{-1} \qquad \boxed{A = PDP^{-1}}$

$$A^m = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\dots(\cancel{P}DP^{-1}) = PD^mP^{-1}$$

$$= \begin{pmatrix} 15 & -1 \\ 70 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0.15 \end{pmatrix}^m \cdot \underbrace{\frac{1}{85}\begin{pmatrix} 1 & 1 \\ -70 & 15 \end{pmatrix}}_{P^{-1}}$$

$$\to \begin{pmatrix} 15 & -1 \\ 70 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{85}\begin{pmatrix} 1 & 1 \\ -70 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & 0 \\ 70 & 0 \end{pmatrix}\frac{1}{85}\begin{pmatrix} 1 & 1 \\ -70 & 15 \end{pmatrix} = \frac{1}{85}\begin{pmatrix} 15 & 15 \\ 70 & 70 \end{pmatrix}$$

$$= \begin{pmatrix} 15/85 & 15/85 \\ 70/85 & 70/85 \end{pmatrix}$$

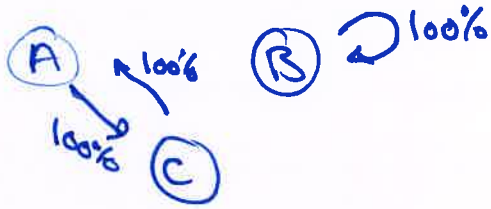
$$A^n \to \begin{pmatrix} 15/85 & 15/85 \\ 70/85 & 70/85 \end{pmatrix} \qquad x_0 + y_0 = 1 \qquad \underline{v} = \begin{pmatrix} 15/85 \\ 70/85 \end{pmatrix}$$

$$A^M \cdot \underline{v_0} \Rightarrow \begin{pmatrix} 15/85 & 15/85 \\ 70/85 & 70/85 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = x_0 \cdot \underline{v} + y_0 \cdot \underline{v} = \underline{v}$$

$$= \begin{pmatrix} 15/85 \\ 70/85 \end{pmatrix}$$

Equilibrium state of the Markov chain.

d) $$A = \begin{array}{c} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \begin{matrix} A & B & C \end{matrix} \end{array}$$



$$\underline{v_0} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \to \begin{pmatrix} z_0 \\ y_0 \\ x_0 \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \to \cdots$$

<u>No equilibrium state</u> (except if $x_0 = z_0 = 0,\ y_0 = 1$)

② Problems from [E].

3.15 $A\underline{x}=\underline{b}$ lin. system

Assume: $x_1 \neq x_2$ solutions $\boxed{\begin{array}{l} A\cdot \underline{x_1} = \underline{b} \\ A\cdot x_2 = \underline{b}\end{array}}$

Prove: $\lambda x_1 + (1-\lambda)\underline{x_2}$ is a solution for all λ.

$\Downarrow$

Compute: $A\cdot(\lambda\underline{x_1}+(1-\lambda)\underline{x_2})$ ⟵ must show that this is $\underline{b}$

$= A\cdot\lambda\cdot\underline{x_1} + A\cdot(1-\lambda)\underline{x_2}$

$= \lambda \underbrace{A x_1} + (1-\lambda)\underbrace{A x_2}$

$= \lambda\cdot\underline{b} + (1-\lambda)\underline{b}$

$= \cancel{\lambda \underline{b}} + \underline{b} - \cancel{\lambda\underline{b}} = \underline{b}$ ok

3.13 b)

$$A = \begin{pmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{pmatrix}$$

$\boxed{\begin{array}{c} |A| = 0 \\ \Updownarrow \\ \text{rk } A < 3 \end{array}}$

$|A| = (t+3)\cdot((t-3)(t+4)+6)$

$\quad -5(-(t+4)+6) + 6(-1-(t-3))$

$= (t+3)((t-3)(t+4)+6) + \underbrace{5t - 6t - 10 + 12}_{-t+2 = -(t-2)}$

$= (t+3)(\underbrace{t^2+t-6}_{(t-2)(t+3)}) - \underline{(t-2)}$

$= (t-2)\cdot((t+3)^2 - 1)$

$= (t-2)(t^2+6t+8) = (t-2)(t+2)(t+4) = 0$

$t=2,\ t=-2,\ t=-4$

$t \neq -2, 2, -4$: $\operatorname{rk} A = 3$

$$A = \begin{pmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{pmatrix}$$

$t = -2$: $\begin{pmatrix} 1 & 5 & 6 \\ -1 & -5 & -6 \\ 1 & 1 & 2 \end{pmatrix}$ $\Rightarrow \operatorname{rk} A = 2$

$\neq 0$

$t = 2$: $\begin{pmatrix} 5 & 5 & 6 \\ -1 & -1 & -6 \\ 1 & 1 & 6 \end{pmatrix}$ $\neq 0$ $\Rightarrow \operatorname{rk} A = 2$

$t = -4$: $\begin{pmatrix} -1 & 5 & 6 \\ -1 & -7 & -6 \\ 1 & 1 & 0 \end{pmatrix}$ $\Rightarrow \operatorname{rk} A = 2$

$\neq 0$

$$\operatorname{rk} A = \begin{cases} 2, & t = 2, -2, -4 \\ 3, & \text{otherwise} \end{cases}$$

2.11. $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r\}$ r vectors in $\mathbb{R}^n$

Show: $r > n \Rightarrow$ the vectors are linearly ~~dependent~~ dependent

$$A = \underbrace{\left(\underline{v}_1 \,\middle|\, \underline{v}_2 \,\middle|\, \underline{v}_3 \,\middle|\, \ldots \,\middle|\, \underline{v}_r \right)}_{r} \Big\} n$$

$\operatorname{rk} A \leq n < r$

$\Downarrow$

$\operatorname{rk} A < r$

There are column without pivot $\Rightarrow$ linearly dependent

$$\underbrace{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}_{4} \Big\} 3$$