Explain: If $r=2$ is a double root in the char. eqn. of a second order homog. diff. eqn. then the general solution is $y = c_1 e^{2t} + c_2 \cdot t e^{2t}$.

Answer: We consider $y'' + ay' + by = 0$ with char. eqn. $r^2 + ar + b = 0$, and assume that $r_1 = r_2 = 2$. Then: $\underline{y = e^{2t}}$ and $y = \underline{t e^{2t}}$ are solutions

$(r-2)^2 = 0$

$r^2 - 4r + 4 = 0$

i) Check that $\underline{y = e^{2t}}$ is a solution of $y'' - 4y' + 4y = 0$

ii) — '' — $\underline{y = t e^{2t}}$ — '' —

Conclusion: If y_1 and y_2 are solutions, then $c_1 y_1 + c_2 y_2$ has to be a solution for all c_1, c_2 because the eqn. is linear.

$\Rightarrow$ this is the general solution.

$$A = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix} :$$

Eq. state: A state vector $\underline{v}$ (v_1, v_2, v_3) such that $A \cdot \underline{v} = \underline{0}$

$\Uparrow$

$A\underline{v} - I\underline{v} = 0$

$(A - 1 \cdot I)\underline{v} = 0$

and $v_1 + v_2 + v_3 = 1$.

nullsp. of eigenvectors with $\lambda = 1$.

$\lambda = 1$:

$$\begin{pmatrix} -0.25 & 0.02 & 0.10 \\ 0.20 & -0.10 & 0.20 \\ 0.05 & 0.08 & -0.30 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -0.05 & -0.08 & 0.30 \\ 0.20 & -0.10 & 0.20 \\ 0.05 & 0.08 & -0.30 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -0.05 & -0.08 & 0.30 \\ 0 & -0.42 & 1.40 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-0.42 v_2 + 1.40 v_3 = 0$$

$$v_2 = \frac{1.40}{0.42} v_3 = \frac{10}{3} v_3$$

$$-0.05 v_1 - 0.08 v_2 + 0.3 v_3 = 0$$

$$-0.05 v_1 = 0.08 \left(\frac{10}{3} v_3 \right) - 0.3 v_3$$

$$-0.05 v_1 = \frac{8}{3} v_3 - \frac{3}{10} v_3$$

$$v_1 = \left(\frac{-160}{3} + 6 \right) v_3 = \frac{-142}{3} v_3$$

use $v_1 + v_2 + v_3 = 1$ to find eq. state $\underline{v} = (v_1, v_2, v_3)$

Problemset 8, 3b: $\max\ xz + yw$ when $\begin{cases} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{cases}$

$$\mathcal{L} = xz + yw - \lambda_1 (x^2+y^2) - \lambda_2 (4z^2+9w^2)$$

FOC:

$\mathcal{L}'_x = z - \lambda_1 \cdot 2x = 0$

$\mathcal{L}'_y = w - \lambda_1 \cdot 2y = 0$

$\mathcal{L}'_z = x - \lambda_2 \cdot 8z = 0$

$\mathcal{L}'_w = y - \lambda_2 \cdot 18w = 0$

C: $x^2+y^2 \le 1$

$4z^2+9w^2 \le 36$

CSC: $\lambda_1 \ge 0$, $\lambda_2 \ge 0$

$\lambda_1(x^2+y^2-1) = 0$

$\lambda_2(4z^2+9w^2-36) = 0$

Alt 1: Constraints

a	b	c	d
$g_1 = 1$, $g_2 = 36$	$g_1 < 1$, $g_2 = 36$	$g_1 = 1$, $g_2 < 36$	$g_1 < 1$, $g_2 < 36$

Alt 2:

A	B	C	D
$\lambda_1 = 0$, $\lambda_2 = 0$	$\lambda_1 = 0$, $\lambda_2 > 0$	$\lambda_1 > 0$, $\lambda_2 = 0$	$\lambda_1 > 0$, $\lambda_2 > 0$

Alt 3 using FOC's

$z = 2\lambda_1 x \Rightarrow x - 8\lambda_2 \cdot (2\lambda_1 x) = 0$ $\quad x(1 - 16\lambda_1\lambda_2) = 0$

$x = 0$ or $\lambda_1\lambda_2 = 1/16$

$w = 2\lambda_1 y \Rightarrow y - 18\lambda_2 (2\lambda_1 y) = 0$ $\quad y(1 - 36\lambda_1\lambda_2) = 0$

$y = 0$ or $\lambda_1\lambda_2 = 1/36$

Cases:

(i)	(ii)	(iii)
$x = 0$, $y = 0$	$x = 0$, $\lambda_1\lambda_2 = 1/36$	$\lambda_1\lambda_2 = 1/16$, $y = 0$

i) $x = 0,\ y = 0$: $\quad \lambda_1 = 0$, $\lambda_2 = 0$ $\Rightarrow$ $(0,0,0,0;0,0)$

$z = 0,\ w = 0$

ii) $x = 0,\ \lambda_1\lambda_2 = 1/36$: $\quad \lambda_1 > 0,\ \lambda_2 > 0$

$z = 0$, $w = 2\lambda_1 y$ $\quad x^2+y^2 = 1$ $\quad y^2 = 1$ $\quad y = \pm 1$

$4z^2+9w^2 = 36$ $\quad 9w^2 = 36$ $\quad w = \pm 2$

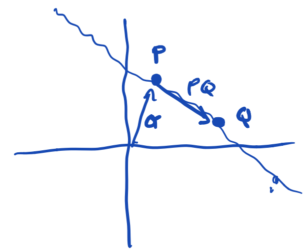
$\pm 2 = 2\lambda_1 \cdot (\pm 1) \Rightarrow \lambda_1 = 1 \Rightarrow$ $(0,1,0,2;\ 1,1/36)$

$\lambda_2 = 1/36$ $\quad (0,-1,0,-2;\ 1,1/36)$

iii) Similar to ii)

[E] Chap 2 :

2.5. b) $\ell: 2x+3y=1$



$x=2: \quad P=(2,-1)$
$x=0: \quad Q=(0,1/3)$

$PQ=(-2,4/3)$

$(x,y) = OP + t \cdot PQ$
$= (2,-1)+t\cdot(-2,4/3)$
$= (2-2t, -1+4t/3)$

2.8 a) $\underline{v} = (-1,2)$
$\underline{w} = (3,-6)$

$\underline{w} = -3\underline{v}$
lin dependent

b) $\underline{v} = (2,-1)$
$\underline{w} = (3,4)$

$(3,4) = c\cdot(2,-1)$
$3=2c$
$4=-c$
not possible $\Rightarrow$ lin independent

$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad b \neq 0$

$\lambda_1=2, \lambda_2=3: \quad (\lambda-2)(\lambda-3) = \lambda^2 - \underset{\uparrow \atop tr(A)}{5}\lambda + \underset{\uparrow \atop det(A)}{6}$

$$\boxed{\begin{aligned} a+c &= 5 \\ a\cdot c - b^2 &= 6 \end{aligned}}$$

Exam 11/2019, 3b: $(2t+2ty^2)+(2y+2yt^2)y'=0$

$$\frac{\cancel{2y(1+t^2)}\,y'}{\cancel{2y(1+t^2)}} = -\frac{2t(1+y^2)}{2y(1+t^2)}$$

$$y' = \underbrace{-\frac{t}{1+t^2}}_{f(t)}\cdot\underbrace{\frac{1+y^2}{y}}_{g(y)}$$

$$\frac{y}{1+y^2}\cdot y' = -\frac{t}{1+t^2}$$

$$\int \frac{y}{1+y^2}\,dy = \int -\frac{t}{1+t^2}\,dt$$

$$\frac{1}{2}\cdot\ln(1+y^2) = -\frac{1}{2}\ln|1+t^2| + C$$

$$u = 1+y^2$$
$$du = 2y\cdot dy$$
$$dy = \frac{du}{2y}$$

$$\ln(1+y^2) = -\ln(1+t^2) + 2C$$

$$1+y^2 = e^{-\ln(1+t^2)+2C} = \frac{e^{2C}}{1+t^2} = \frac{K}{1+t^2}$$

$$y^2 = \frac{K}{1+t^2} - 1$$

$$y = \pm\sqrt{\frac{K}{1+t^2} - 1}$$

Explain that two non-zero eigenvectors of a matrix with different eigenvalues are linearly independent.

① We assume that $\underline{u}, \underline{w} \neq \underline{0}$ such that $A \cdot \underline{u} = \lambda_1 \cdot \underline{u}$

with $\lambda_1 \neq \lambda_2$. $A \cdot \underline{w} = \lambda_2 \cdot \underline{w}$

② We want to prove that $\{\underline{u}, \underline{w}\}$ linearly independent, that is, that $\boxed{c_1 \underline{v} + c_2 \underline{w} = \underline{0} \Rightarrow c_1 = c_2 = 0}$

$$c_1 \underline{u} + c_2 \underline{w} = \underline{0}$$
$$A c_1 \underline{u} + A c_2 \underline{w} = A \cdot \underline{0}$$
$$c_1 A\underline{u} + c_2 A\underline{w} = \underline{0}$$
$$c_1 \lambda_1 \underline{u} + c_2 \lambda_2 \underline{w} = \underline{0}$$

$c_2 \underline{w} = -c_1 \underline{u}$ gives

$$c_1 \lambda_1 \underline{u} + (-c_1 \underline{u}) \lambda_2 = \underline{0}$$
$$c_1 \underline{u} \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} = \underline{0}$$
$$c_1 \underbrace{\underline{u}}_{\underline{v} \neq \underline{0}} = \underline{0} \Rightarrow c_1 = 0$$

$$c_2 \underbrace{\underline{w}}_{\underline{w} \neq \underline{0}} = \underline{0}$$

$$c_2 = 0$$

Conclusion: ① implies that $\underline{u}, \underline{w}$ are lin. independent

How to compute A^m and Markov chains

Assume A is diagonalizable:

$$P^{-1} A P = D \quad | P \cdot \qquad P = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$$
$$AP = PD \quad | \cdot P^{-1} \qquad D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$
$$A = PDP^{-1}$$

$$A^m = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}}) \dots (\cancel{P}DP^{-1})$$
$$= PD^m P^{-1} = P \cdot \begin{pmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{pmatrix} P^{-1}$$

Markov chains:

$$\underline{v}_{t+1} = A \cdot \underline{u}_t$$
$$\underline{v}_m = A^m \underline{u}_0$$
$$\downarrow \text{ as } m \to \infty$$
$$(\underline{v} | \underline{v} | \dots | \underline{v})$$

Theory: If the Markov chain is regular, then the eq. state is the unique state vector $\underline{u}$ that is in E_1 (that is, an eigenvector $\lambda = 1$)

Exam 01/2017, Pb 3

$$f(x,y,z) = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2$$

a) $H(f) = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix}$ $\quad D_1 = -4$, $D_2 = 12$, $D_3 = 0$

(RRC): neg. semidefinite

$\Rightarrow f$ concave

b) $f_{max} = -3$ with $(x,y,z) = \dots$

c) $g(w) = 6/w$ with $w = f(x,y,z)$

Composite fn: inner fn $f(x,y,z)$

outer fn $g(w) = 6/w$

$$g(x,y,z) = \frac{6}{-3 - 2x^2 + \dots - 2z^2}$$

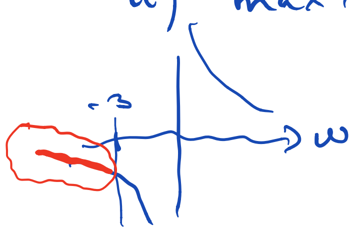
---

i) $w \leq -3$ since $w = f(x,y,z)$, $f_{max} = -3$

ii) max/min $g(w) = 6/w$, $w \leq -3$

min: $g(-3) = 6/-3 = -2$

max: no max

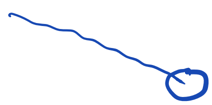


$g(w) = 6/w \quad , \quad w \leq -3$

$g'(w) = (6w^{-1})' = 6 \cdot (-1)\, w^{-2}$

$= \dfrac{-6}{w^2}$

$g'(w)$: -3, 0 ; $-$ for $w < -3$



$\underline{g_{min}} = -2$

no $\underline{g_{max}}$

$$\underline{y}_{t+1} = A \cdot \underline{y}_t \longrightarrow \boxed{\underline{z}_{t+1} = D \cdot \underline{z}_t}$$

General Solution: $\underline{y}_t = c_1 \cdot \underline{v}_1 \lambda_1^t + c_2 \underline{v}_2 \lambda_2^t + \dots$

$$z_{1,t+1} = \lambda_1 \cdot z_{1,t}$$

$$z_{2,t+1} = \lambda_2 \cdot z_{2,t}$$

$$\vdots$$

$$z_{n,t+1} = \lambda_n \cdot z_{n,t}$$

$$z_{1,1} = \overset{0}{\overset{\shortparallel}{\lambda_1}} \cdot z_{1,0} = 0$$

$$z_{1,2} = \lambda_1 \cdot z_{1,m} = \lambda_1 (\lambda_1 z_{1,0}) = \overset{0}{\overset{\shortparallel}{\lambda_1^2}} z_{1,0} = 0$$

$$\vdots$$

$$z_{1,m} = z_{1,0} \cdot \lambda_1^m = \begin{cases} z_{1,0}, & m = 0 \\ 0, & m > 0 \end{cases}$$

when $\lambda_1 = 0$

$\boxed{0^0 = 1 \text{ in this context}}$

Exam 12/2015, 2c:

$h'_t + h'_y \cdot y' = 0$

$(4yt + 4t^3 + 2t) + (2y - 1 + 2t^2)y' = 0$

$h'_t = 4yt + 4t^3 + 2t$

$h'_y = 2y - 1 + 2t^2$

$\Downarrow$

$h = y^2 - y + 2t^2 y + \underbrace{t^4 + t^2}_{\varphi(t)}$ ✓

$\Downarrow$

General solution:

$y^2 - y + 2t^2 y + t^4 + t^2 = C$

$y^2 + (2t^2 - 1)y + (t^4 + t^2 - C) = 0$

$$y = \frac{-(2t^2-1) \pm \sqrt{(2t^2-1)^2 - 4 \cdot 1 \cdot (t^4 + t^2 - C)}}{2}$$

$$y' - y\ln t = y$$

$$y' - y\ln t - y = 0$$

$$y' - (\ln t + 1)y = 0$$

homog. linear eqn.

<u>integrating factor</u>

$$y' = y + y\ln t$$

$$= y \cdot (1 + \ln t)$$

$$\frac{1}{y} y' = 1 + \ln t$$

$$\ln|y| = \int 1 + \ln t \, dt$$

$f = x^2 + y^2 + z^2 + w^2 + \underline{2xz} - \underline{2yw}$ quadratic form

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$H(f) = 2A$

Symmetric matrix

$D_1 = 1$

$D_2 = 1$

$D_3 = 0$

$D_4 = 0$

Method I: RRC

$rk\, A = 2$

$D_1 > 0$, $D_2 > 0$

pos. semi defn.

Stationary pts:

$2A \cdot \underline{x} = \underline{0}$

Max/min:

f convex

global min

Method 2:

$\Delta_1 = 1, 1, 1, 1 \geq 0$

$\Delta_2 = 1, 0, 1, 1, 0, 1 \geq 0$

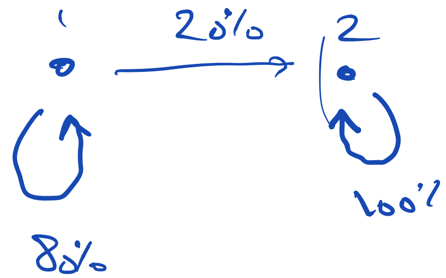
$\Delta_3 = 0, 0, 0, 0 \geq 0$

$\Delta_4 = 0 \geq 0$

pos. semidefn.

Non-regular Markov chains

$$A = \begin{pmatrix} 0.8 & 0 \\ 0.2 & 1 \end{pmatrix}$$



not regular

In general:

$A = (a_{ij})$

all $a_{ij} > 0$: regular

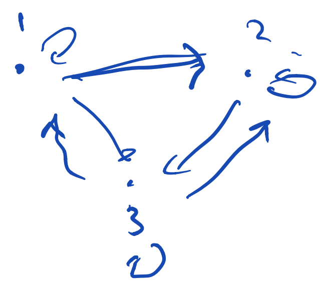
regular $\Longleftrightarrow$ there is a path from any node i to any other node j

Long term state:

$A^m \cdot \underline{v_0}$

$$A = \begin{pmatrix} 0.8 & 0 & 0.05 \\ 0.1 & 0.7 & 0.05 \\ 0.1 & 0.3 & 0.9 \end{pmatrix}$$

regular



Operaters and superposition

Ex: $y'' - 7y' + 12y = t$

$$\boxed{D = \frac{d^2}{dt^2} - 7\frac{d}{dt} + 12}$$

$$D(t^2) = (t^2)'' - 7(t^2)' + 12(t^2)$$
$$= \underline{2 - 14t + 12t^2}$$

All soln' of

y_h: $y'' - 7y' + 12y = 0 \Longleftrightarrow \{\underline{D(y) = 0}\}$

Answer: $\underline{y = C_1 e^{3t} + C_2 e^{4t}}$

one sol.

y_p: $y'' - 7y' + 12y = t \Longleftrightarrow \boxed{D(y) = t}$

Claim: $y_h + y_p$ is the general solution of $y'' - 7y' + 12y = t \Longleftrightarrow$ All sol'n $\boxed{D(y) = t}$

$$\left.\begin{aligned} D(y_h) &= 0 \\ D(y_p) &= t \end{aligned}\right\} \quad \begin{aligned} D(y_h + y_p) &= D(y_h) + D(y_p) \\ &= 0 + t = t \;// \end{aligned}$$

Assume $D(y) = t$. Then we have

$$D(y - y_p) = D(y) - D(y_p)$$
$$= t - t = 0$$

$y - y_p$ is a soln. of $D(*) = 0$

$\parallel$

$y - y_p = y_n \Rightarrow \underline{y = y_n + y_p}.$

Actual Solution:

$$y = y_n + y_p$$
$$= \underline{C_1 e^{3t} + C_2 e^{4t} + \left(\frac{1}{12}t + \frac{7}{144}\right)}$$

y_n: $\quad r^2 - 7r + 12 = 0$

$r = 3, r = 4 \quad \Rightarrow y_n = \underline{C_1 e^{3t} + C_2 e^{4t}}$

y_p: $\quad y'' - 7y' + 12y = t$

$f(t) = t$
$f' = 1$
$f'' = 0$

$y = At + B$
$y' = A$
$y'' = 0$

$0 - 7A + \underline{12(At + B)} = \underline{t}$

$(12A)t + (12B - 7A)$

$12A = 1$, $\quad A = \frac{1}{12}$ $\qquad 12B - 7A = 0$, $\quad 12B = 7 \cdot \frac{1}{12}$ $\qquad B = 7/144$

"Conseptual problems":

By this, I just mean a problem where you don't use a standard algorithm.

Ex:

1. Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Show that $\underline{v}$ is an eigenvector of A, and find the eigenvalue.

Solution: Don't compute all eigenvalues and eigenvectors by standard algorithm.

Instead: Use defn. $A\underline{v} = \lambda \underline{v}$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Concl: Eigenvector with $\lambda = 5$.

2. Consider $y'' - 2y' - 3y = e^{at}$.

We try to find a particular solution of the form $y = A \cdot e^{at}$.

For which values of a is this possible?

Solution: Do not try to solve the equation using superposition.

Instead: Try to put in $y = Ae^{at}$.

$$\left.\begin{aligned} y &= Ae^{at} \\ y' &= A \cdot a e^{at} \\ y'' &= A \cdot a^2 e^{at} \end{aligned}\right\} \quad \begin{aligned} &Aa^2e^{at} - 2Aae^{at} \\ &-3Ae^{at} = e^{at} \end{aligned}$$

$$(Aa^2 - 2Aa - 3A)e^{at} = e^{at}$$

$$Aa^2 - 2Aa - 3A = 1$$

$$A(a^2 - 2a - 3) = 1$$

Concl: Possible for $a \neq 3, -1$ $\longrightarrow$ $A = \dfrac{1}{a^2 - 2a - 3} = \dfrac{1}{(a-3)(a+1)}$

3. Consider $f(x,y,z,w) = e^{x^2+y^2+z^2+w^2+2xw}$

Find the minimum value of f, if it exists.

Solution: Don't compute the Hessian of f!

Instead: $u = x^2+y^2+z^2+w^2+2xw$

quadratic form with symm. matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$D_1 = 1$
$D_2 = 1$
$D_3 = 1$
$D_4 = |A| = 0$

$$|A| = 1 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1 \cdot 1 - 1 \cdot 1 = 0$$

RRC:

$rk = 3$

pos. semidefin

global min. for u:

$u_{min} = u(0,0,0,0) = \underline{0}$

Outer fn: $h(u) = e^u$

increasing since $h'(u) = e^u > 0$

$\Rightarrow f_{min} = e^0 = \underline{\underline{1}}$