Plan

1. Envelope theorem: Unconstrained case
2. Envelope theorem: Constrained case
3. Lagrange multipliers

(a) Review: integration FORK1003 Lecture 4 / [FT] Appendix B

(b) Exams: GRA6035 / ELE3781 school exam Nov 26th / A–F / oral exams for some students

(c) Bordered Hessians: not curriculum this year

Review:

a) Second order condition: (SOC)

If $(\underline{x}^*; \lambda^*)$ satisfies FOC+C/FOC+C+CSC in a Lagrange/Kuhn-Tucker problem, then:

$h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*)$ concave $\Rightarrow$ $\underline{x}^*$ max

— " — convex $\Rightarrow$ $\underline{x}^*$ min

b) Nondegenerate constraint qualification: (NDCQ)

Constraints in Lagrange case:

$$\boxed{\begin{matrix} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{matrix}} \rightsquigarrow J = \begin{pmatrix} \partial g_1/\partial x_1 & \cdots & \partial g_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial g_m/\partial x_1 & \cdots & \partial g_m/\partial x_n \end{pmatrix}$$

NDCQ: rk J maximal

Admissible point where NDCQ fails: possible max/min

Kuhn-Tucker case:

As in the Lagrange case, but we only consider binding constraints (hold with =)

Method:

i) Find candidate pts

a) ordinary: Solve FOC+C / FOC+C+CSC

b) exceptional: C holds + NDCQ fails

‖

admissible

ii) Check if max/min

c) EVT: D bounded $\Rightarrow$ max/min

d) SOC:

① Envelope theorems : Unconstrained case

Ex: max $f(x) = 1 + 2x - x^2$

$f'(x) = 2 - 2x = 0$

$x = 1$ ← stationary pts

$f''(x) = -2 < 0$ for all x ⇒ f concave

$x^* = 1$ max point $\quad f_{max} = f^* = f(1) = 2$

$x^*(a) \qquad \boxed{\begin{array}{l} x^*(2) = 1 \\ f^*(2) = 2 \end{array}}$ } $a = 2$ special case

$f^*(a)$

max $f(x) = 1 + ax - x^2$ (a parameter)

$f'(x) = a - 2x = 0$

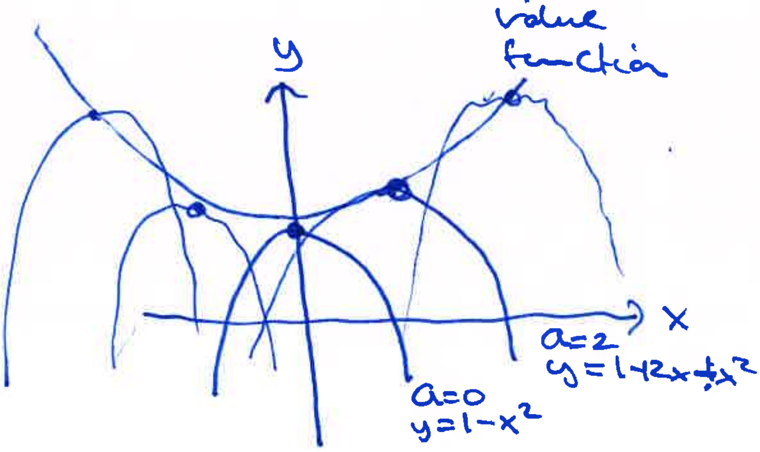
$x = a/2$ ← stationary points

$f''(x) = -2 < 0$ ⇒ f concave

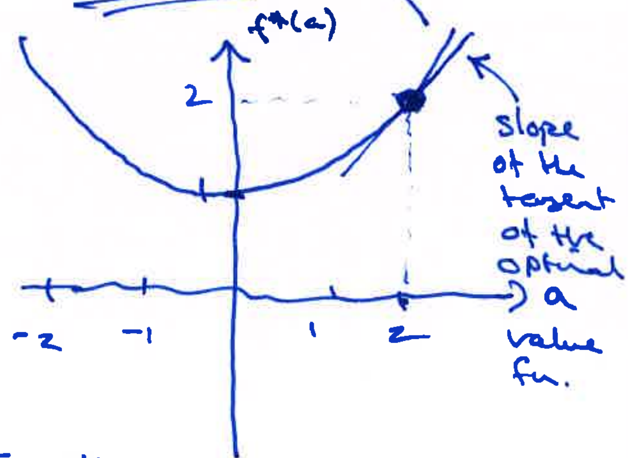
$x^*(a) = a/2$ max pt. ⇒ $f^*(a) = 1 + a \cdot (a/2) - (a/2)^2$

$= 1 + a^2/2 - a^2/4$

optimal value function → $= 1 + a^2/4$



$f(x) = 1 + ax - x^2$



Env. thm

$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(x^*(a))$$

In this case: $x^*(2) = 1$ $\quad f = 1 + ax - x^2$

$f^*(2) = 2$

$\frac{\partial f}{\partial a} = x \qquad \frac{\partial f}{\partial a}(x^*(a)) = x^*(a) = ?$

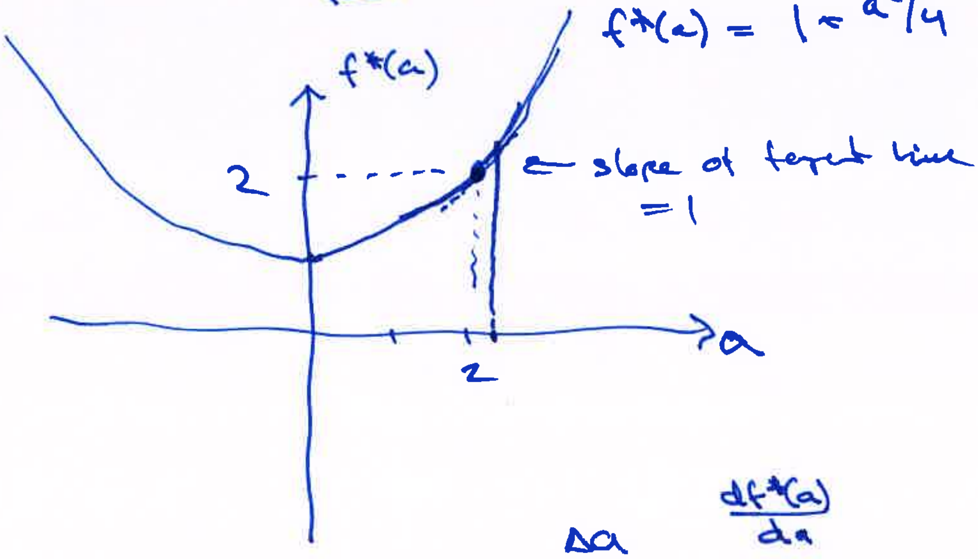
a=2: $x^*(a) = 1 \Rightarrow \frac{\partial f}{\partial a}(x^*(a)) = 1$ at $a=2$

Env. thm: $\boxed{\frac{df^*(a)}{da} = \underbrace{\frac{\partial f}{\partial a}(x^*(a))}}$ $= 1$ at $a=2$

slope of the tangent of the optimal value fn.

Short cut to compute the slope of this tangent

$f^*(a) = 1 + a^2/4$



$$f^*(2{,}1) \cong f^*(2) + \underset{\substack{\| \\ 0.1}}{\overset{\substack{\Delta a \\ \|}}{(2{,}1-2)}} \cdot \overset{\frac{df^*(a)}{da}}{1} = 2 + 0.1 \cdot 1 = \underline{2.1}$$

Formula: $f^*(a) \simeq f^*(2) + \underset{\substack{\| \\ (a-2)}}{\Delta a} \cdot \frac{\partial f^*(a)}{da}$ when a is close to 2

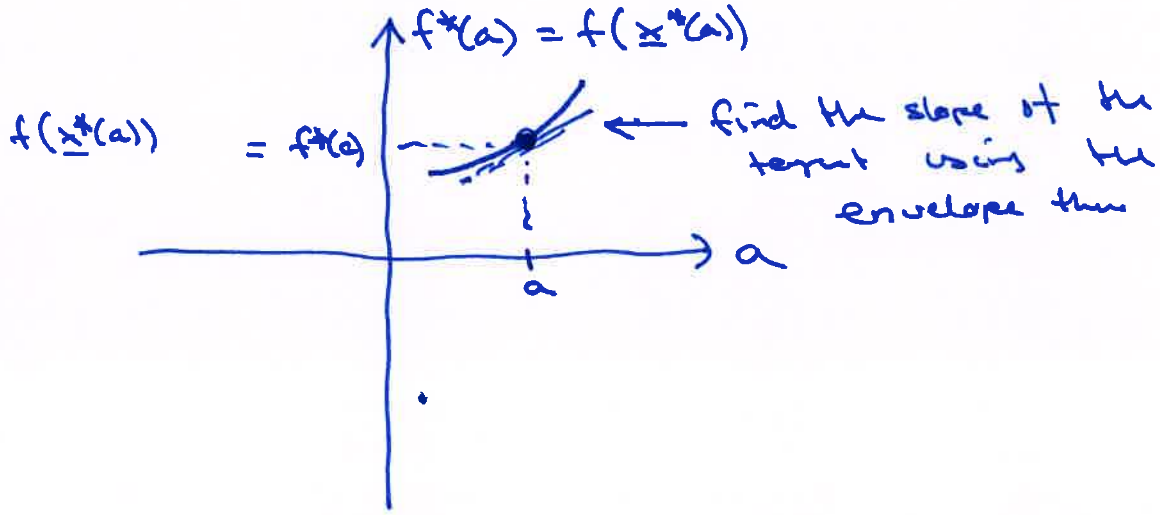
Envelope theorem: Unconstrained case

Assume that $\underline{x}^*(a)$ is the solution of the maximal / minimum value problem

$$\max/\min \ f(\underline{x};a)$$

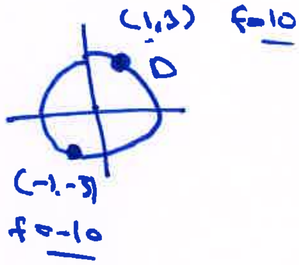
Then:

$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(\underline{x}^*(a))$$



② Constrained case: Envelope theorem

Ex: max/min $f(x,y) = x + 3y$ when $x^2 + y^2 = 10$



D Bounded $\underset{\text{EVT}}{\Rightarrow}$ there is max/min

ordinary cand. pts:

$\mathcal{L} = x + 3y - \lambda(x^2 + y^2)$

FOC $\begin{cases} \mathcal{L}'_x = 1 - \lambda \cdot 2x = 0 \\ \mathcal{L}'_y = 3 - \lambda \cdot 2y = 0 \end{cases}$ $\quad x = \frac{1}{2\lambda} \quad (\lambda \neq 0)$, $\quad y = \frac{3}{2\lambda}$

C $\quad x^2 + y^2 = 10$

$\Downarrow$

$x^2 + y^2 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$

$(2\lambda)^2 = 1 \quad \Leftarrow \quad \frac{10}{(2\lambda)^2} = 10 \quad \Leftarrow \quad \frac{1^2 + 3^2}{(2\lambda)^2} = 10$

$2\lambda = \pm\sqrt{1} = \pm 1$

$\lambda = \underline{\pm 1/2} \quad \rightarrow \quad (x,y;\lambda) = \underset{f=\underline{10}}{\underline{(1,3;1/2)}}, \quad \underset{f=\underline{-10}}{\underline{(-1,-3;-1/2)}}$

NDCQ: $\quad J = (2x \quad 2y)$

$x^2 + y^2 = 10$

NDCQ fails:

$\text{rk}\,(2x \quad 2y) < 1$

$2x = 2y = 0 \Rightarrow \underline{(0,0)}$

But $0^2 + 0^2 = 0 \neq 1$

not adm.

$f_{max} = \underline{10}$ at $(x,y) = (1,3)$ with $\lambda = 1/2$

$f_{min} = \underline{-10}$ at $(x,y) = (-1,-3)$ with $\lambda = -1/2$

max $f(x,y) = x + 3y$ when $x^2 + y^2 = 10$

Found: $f^* = 10$ $\quad (x^*, y^*; \lambda^*) = (1, 3; 1/2)$

What if: i) $\boxed{\max\ x + 4y \text{ when } x^2+y^2 = 10}$

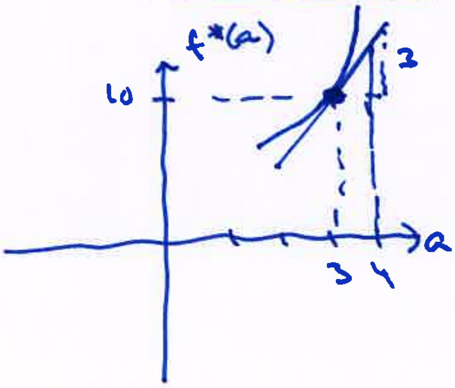
ii) max $x + 3y$ when $x^2 + y^2 = 11$

i) $\boxed{\max\ f(x,y) = x + ay \text{ when } x^2+y^2=10}$

a=3: $x^*(3) = \underline{1}$, $y^*(3) = \underline{3}$, $\lambda^*(3) = \underline{1/2}$, $f^*(3) = \underline{10}$

Envelope thm: $\dfrac{df^*(a)}{da} = L'_a(x^*(a), y^*(a); \lambda^*(a))$

$$L'_a = \frac{\partial L}{\partial a}$$



$\dfrac{\partial L}{\partial a} = L'_a = y \Rightarrow \dfrac{df^*(a)}{da} = y^*(a)$

at a=3: $= y^*(3)$

$= \underline{\underline{3}}$

$L = x + ay - \lambda(x^2+y^2-10)$

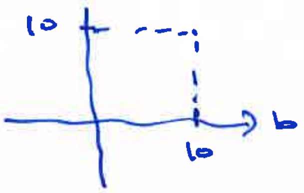
Conclusion: $f^*(4) \simeq f^*(3) + \underbrace{(4-3)}_{\Delta a} \cdot \dfrac{df^*(a)}{da}$

$= 10 + 1 \cdot 3 = \underline{\underline{13}}$

ii) max $f(x,y) = x + 3y$ when $x^2 + y^2 = b \Rightarrow x^2 + y^2 - b = 0$

$L = x + 3y - \lambda(x^2+y^2-b) = x + 3y - \lambda x^2 - \lambda y^2 + \lambda b$

b=10: $x^*(10) = 1$, $y^*(10) = 3$, $\lambda^*_{10} = 1/2$, $f^*(10) = 10$



$f^*(11) \simeq f^*(10) + \underbrace{(11-10)}_{\Delta b} \cdot \dfrac{df^*(b)}{db} \leftarrow L'_b(x^*(b), y^*(b); \lambda^*(b))$

$= 10 + 1 \cdot 1/2$ $\qquad = \lambda^*(b)$

$= \underline{\underline{10.5}}$ $\qquad = \lambda^*(10) = 1/2$

Envelope Thm: Constrained case   Lagrange / Kuhn-Tucker case

Assume that $(\underline{x}^*(a);\underline{\lambda}^*(a))$ is an optimal point (max/min) in a Lagrange / Kuhn-Tucker problem with parameter a

$$\text{max/min } f(\underline{x}) \text{ when } \begin{cases} g_1(\underline{x})-a_1=0 \\ \vdots \\ g_m(\underline{x})-a_m=0 \end{cases} \quad \text{or} \quad \begin{cases} g_1(\underline{x})-a_1\le 0 \\ \vdots \\ g_m(\underline{x})-a_m\le 0 \end{cases}$$

and that $(\underline{x}^*(a);\underline{\lambda}^*(a))$ satisfy FOC+C / FOC+C+CSC.

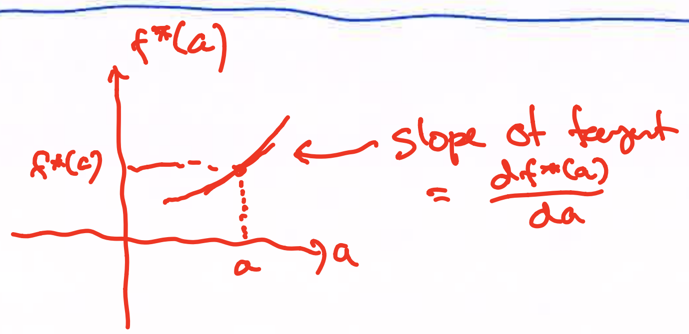
Then:

$$\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}(\underline{x}^*(a);\underline{\lambda}^*(a))$$

where $L = f(\underline{x}) - \lambda_1(g_1(\underline{x})-a_1) - \lambda_2(g_2(\underline{x})-a_2) - \dots - \lambda_m(g_m(\underline{x})-a_m)$

and $f^*(a) = f(\underline{x}^*(a))$.

$f(\underline{x}) - \lambda_1 g_1(\underline{x}) + \boxed{\lambda_1 a_1} - \lambda_2 g_2(\underline{x}) + \boxed{\lambda_2 a_2} - \dots$



$$\frac{\partial L}{\partial a_i} = \lambda_i$$

$$\parallel$$

$$\frac{df^*(a_i)}{da_i} = \lambda_i^*(a_i)$$

③ Lagrange multipliers: $\lambda_1, \lambda_2, \ldots, \lambda_m$

Interpretation: $\lambda_i = \frac{df^*(a)}{da}$ when $a = a_i$ = constant in constraint #i

$$\max f(x,y,z) = x - 2y + z \quad \text{when} \quad \begin{cases} x^2+y^2+z^2 = 10 \\ x+y+z = 3 \end{cases}$$

$\Downarrow$

$(x^*, y^*, z^*; \lambda_1^*, \lambda_2^*)$ max point

Interpretation: $\lambda_1^* \simeq \begin{Bmatrix} \text{max value} \\ \text{if } a_1 = 11 \end{Bmatrix} - \begin{Bmatrix} \text{max value} \\ \text{if } a = 10 \end{Bmatrix}$

"How much the max value changes if you change $x^2+y^2+z^2=10$ to $x^2+y^2+z^2=11$."

$\Downarrow$

Special case of envelope thm

Ex: Kelly's criterion

win $\quad p(\text{win}) = p$

lose $\quad p(\text{lose}) = 1-p = q$

$$\boxed{f(x) = p\ln(1+x) + \underset{1-p}{\overset{\shortparallel}{q}}\cdot\ln(1-x)}, \qquad 0 \leq x < 1$$

Unconstrained opt. problem: $\quad \max f(x)$

$p = \underline{0.60}$: $\quad f'(x) = p\cdot\frac{1}{1+x} + q\cdot\frac{1}{1-x}\cdot(-1)$

$$= \underline{\frac{p}{1+x} - \frac{q}{1-x}} = 0 \qquad | \cdot (1+x)(1-x)$$

$$p(1-x) - q(1+x) = 0$$

$$p - px - q - qx = 0$$

Stationary pt. ✓

$$p - q = (p+q)x = x$$

$$x = p - q = 0.60 - 0.40 = \underline{0.20}$$

$$f''(x) = p\cdot(-1)(1+x)^{-2}\cdot 1 - q\cdot(-1)(1-x)^{-2}\cdot(-1)$$

$$= \frac{-p}{(1+x)^2} - \frac{q}{(1-x)^2} < 0 \Rightarrow f \text{ \underline{concave}}$$

$$x^*(0.60) = \underline{\underline{0.2}} \qquad f^*(0.60) = f(0.20) = 0.6\cdot\ln(1.2) + 0.40\cdot\ln(0.8) \simeq \underline{0.02}$$

$$\frac{df^*(p)}{dp} = \frac{\partial f}{\partial p}(x^*(p)) = \ln(1+x) - \ln(1-x)\Big|_{x = x^*(p)}$$

$$= \ln(1+x^*(p)) - \ln(1-x^*(p))$$

$p = \underline{0.60}$: $\quad = \ln(1.2) - \ln(0.8) \approx \underline{0.41 > 0}$

$p = \underline{0.70}$: $\quad f^*(0.70) \simeq f^*(0.6) + 0.10\cdot 0.41$

$$\simeq 0.02 + 0.04 = \underline{\underline{0.06}}$$

Exploration: Kelly Criterion

X_0 : start capital

B_1 : bet (first game)

X_1 : capital after first game

X_0 — win, $p(\text{win}) = p$: $X_1 = X_0 + B_1$

X_0 — loose, $p(\text{loose}) = q = 1-p$: $X_1 = X_0 - B_1$

$$E(X_1) = p(X_0 + B_1) + (1-p)(X_0 - B_1)$$
$$= X_0 + pB_1 - (1-p)B_1$$
$$= X_0 + (p-q)B_1$$

assume $p - q > 0$ $(p > 0.5)$

$$E(X_1 - X_0) = (p-q)B_1 > 0$$

(expected return)

Play n games, each bet

$$B_i = x \cdot X_{i-1} \quad (0 \le x \le 1)$$

(ratio x of capital).

After 1 game:

W: $X_1 = X_0 + xX_0 = X_0(1+x)$

L: $X_1 = X_0 - xX_0 = X_0(1-x)$

After n games:

$$X_n = X_0 \cdot (1+x)^W \cdot (1-x)^L$$

$$\left.\begin{array}{l} W = \#\text{wins} \\ L = \#\text{looses} \end{array}\right\} W + L = n$$

expected $\left|\begin{array}{l}\text{exponential} \\ \text{growth rate} \\ \text{per game}\end{array}\right.$

$= f(x)$

$$E\left[\tfrac{1}{n}\ln(X_n/X_0)\right] = E\left[\tfrac{1}{n}\ln\left[(1+x)^W \cdot (1-x)^L\right]\right]$$
$$= E\left[\tfrac{1}{n}\left(W \ln(1+x) + L \cdot \ln(1-x)\right)\right]$$
$$= E\left[\tfrac{W}{n}\right]\ln(1+x) + E\left(\tfrac{L}{n}\right) \cdot \ln(1-x)$$
$$= p \ln(1+x) + q \cdot \ln(1-x) = f(x)$$

That means,

$f(x) = g$

$\Downarrow$

"$X_n = X_0 e^{gn}$"

expected growth