Plan

1. Constrained optimization: Second order conditions (SOC)
2. Non-degenerate constraint qualification (NDCQ)

Exams:

Midterm GRA6035Z cancelled.

Mock exam: lecture plan at 13.00

Constrained optimization problem:

Lagrange:

max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Kuhn-Tucker: (std. form)

max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$

Method: ① Find candidate points (for max/min)

(a) Solve FOC+C / FOC+C+CSC (ordinary cand. pts)

(b) Adm. pts (C) that do not satisfy NDCQ (non-degenerate constraint qualification) (exceptional cand. pts)

② Decide if any cand. pt. is max/min

(c) Extreme value theorem (EVT)

D = all adm. pts = all pts satisfying all constraints; compact (bounded) ⇒ there is a max/min.

(d) Second order condition for Lagrange/Kuhn-Tucker problems. (SOC)

Ex: $\min f(x,y,z) = 2x^2+y^2+3z^2$ when $\begin{cases} x-y+2z \geq 3 \\ x+y \geq 3 \end{cases}$

f Std. form: $\cdot(-1)$ $\cdot(-1)$

$-f$ $\boxed{\max -f(x,y,z) = -2x^2-y^2-3z^2 \text{ when } \left|\begin{array}{l} -x+y-2z \leq -3 \\ -x-y \leq -3 \end{array}\right.}$

(a) Find ordinary candidate pts: FOC + C + CSC

$$L = -2x^2-y^2-3z^2-\lambda_1(-x+y-2z)-\lambda_2(-x-y)$$
$$= -2x^2-y^2-3z^2+\lambda_1(x-y+2z)+\lambda_2(x+y)$$

FOC $\begin{cases} L'_x = -4x+\lambda_1+\lambda_2 = 0 \\ L'_y = -2y-\lambda_1+\lambda_2 = 0 \\ L'_z = -6z+2\lambda_1 = 0 \end{cases}$

C: $x-y+2z \geq 3$, $x+y \geq 3$

CSC: $\lambda_1 \geq 0$, $\lambda_1(x-y+2z-3)=0$; $\lambda_2 \geq 0$, $\lambda_2(x+y-3)=0$

i) $x-y+2z=3$, $x+y=3$	ii) $x-y+2z=3$, $x+y>3$	iii) $x-y+2z>3$, $x+y=3$	iv) $x-y+2z>3$, $x+y>3$
$\boxed{\begin{array}{l} -4x+\lambda_1+\lambda_2=0 \\ -2y-\lambda_1+\lambda_2=0 \\ -6z+2\lambda_1=0 \\ x-y+2z=3 \\ x+y=3 \end{array}}$	$\leftarrow *$	$*$	$*$
	$x-y+2z=3$, $x+y>3$	$**$	$**$
$\lambda_1 \geq 0$, $\lambda_2 \geq 0$	$\lambda_1 \geq 0$, $\lambda_2 = 0$	$\lambda_1 = 0$, $\lambda_2 \geq 0$	$\lambda_1 = 0$, $\lambda_2 = 0$

Case i) using Gauss:

$$\begin{aligned}
-4x \quad\quad\quad\quad + \lambda_1 + \lambda_2 &= 0 \\
-2y \quad\quad - \lambda_1 + \lambda_2 &= 0 \\
-6z + 2\lambda_1 \quad\quad &= 0 \\
x - y + 2z \quad\quad\quad &= 3 \\
x + y \quad\quad\quad\quad\quad &= 3
\end{aligned}$$

$$\left(\begin{array}{ccccc|c}
-4 & 0 & 0 & 1 & 1 & 0 \\
0 & -2 & 0 & -1 & 1 & 0 \\
0 & 0 & -6 & 2 & 0 & 0 \\
1 & -1 & 2 & 0 & 0 & 3 \\
1 & 1 & 0 & 0 & 0 & 3
\end{array}\right)$$

$$\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 0 & 3 \\
1 & -1 & 2 & 0 & 0 & 3 \\
-4 & 0 & 0 & 1 & 1 & 0 \\
0 & -2 & 0 & -1 & 1 & 0 \\
0 & 0 & -6 & 2 & 0 & 0
\end{array}\right) \begin{matrix} -1 \\ 4 \end{matrix}
\to
\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 0 & 3 \\
0 & -2 & 2 & 0 & 0 & 0 \\
0 & 4 & 0 & 1 & 1 & 12 \\
0 & -2 & 0 & -1 & 1 & 0 \\
0 & 0 & -6 & 2 & 0 & 0
\end{array}\right) \begin{matrix} 2 \\ -1 \end{matrix}$$

$$\to
\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 0 & 3 \\
0 & -2 & 2 & 0 & 0 & 0 \\
0 & 0 & 4 & 1 & 1 & 12 \\
0 & 0 & -2 & -1 & 1 & 0 \\
0 & 0 & -6 & 2 & 0 & 0
\end{array}\right) \updownarrow$$

$$\to
\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 0 & 3 \\
0 & -2 & 2 & 0 & 0 & 0 \\
0 & 0 & -2 & -1 & 1 & 0 \\
0 & 0 & 4 & 1 & 1 & 12 \\
0 & 0 & -6 & 2 & 0 & 0
\end{array}\right) \begin{matrix} 2 \\ -3 \end{matrix}
\to
\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 0 & 3 \\
0 & -2 & 2 & 0 & 0 & 0 \\
0 & 0 & -2 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 3 & 12 \\
0 & 0 & 0 & 5 & -3 & 0
\end{array}\right) 5$$

$$\rightarrow \begin{array}{c} \begin{array}{ccccc|c} x & y & z & \lambda_1 & \lambda_2 & \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 3 & 12 \\ 0 & 0 & 0 & 0 & 12 & 60 \end{array}\right) \end{array}$$

echelon form

$12\lambda_2 = 60$ $\quad \underline{\lambda_2 = 5}$

$-\lambda_1 + 15 = 12$ $\quad \underline{\lambda_1 = 3}$

$-2z - 3 + 5 = 0$ $\quad \underline{z = 1}$

$-2y + 2z = 0$ $\quad y = 1$

$x + y = 3$ $\quad \underline{x = 2}$

Check: $\lambda_1, \lambda_2 \geq 0$ ok

$\Rightarrow (x,y,z;\lambda_1,\lambda_2) = \underline{(2,1,1;3,5)}$ Cond. pts from case i).

Try to use SOC on this candidate pt:

$(x,y,z;\lambda_1,\lambda_2) = (2,1,1;\underline{3,5}) \rightsquigarrow h(x,y,z) = L(x,y,z;3,5)$

$= -2x^2 - y^2 - 3z^2 + 3(x - y + 2z) + 5(x+y)$

$h'_x = -4x + 3 + 5$

$h'_y = -2y - 3 + 5$

$h'_z = -6z + 6$

Is h concave?

$$H(h) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$D_1 = -4 < 0$
$D_2 = 8 > 0$
$D_3 = -48 < 0$
neg. def.

$\Rightarrow h$ concave
$\Downarrow$ SOC
$(x,y,z) = (2,1,1)$ is max for $-f$.

$(-f)_{max} = -f(2,1,1) = -12$

$(x,y,z) = (2,1,1)$ is min for f.

$f_{min} = f(2,1,1)$
$= 2 \cdot 4 + 1 \cdot 1 + 3 \cdot 1$
$= \underline{\underline{12}}$

at $(x,y,z) = (2,1,1)$ with $\lambda_1 = 3, \lambda_2 = 5$.

$f = 2x^2 + y^2 + 3z^2$

$-f = -2x^2 - y^2 - 3z^2$

Second order condition (SOC) for Lagrange/Kuhn-Tucker pb's

If $(\underline{x}^*; \underline{\lambda}^*)$ is a candidate pt satisfying FOC+C / FOC+C+CSC, then:

$$\boxed{\begin{array}{l} h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*) \text{ convex} \Rightarrow \underline{x}^* \text{ is a min} \\ h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*) \text{ concave} \Rightarrow \underline{x}^* \text{ is a max} \end{array}}$$

__Remark:__ if $h(\underline{x})$ is **not** convex/concave, $\underline{x}^*$ could still be min/max

② **NDCQ:**

i) Lagrange case: $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Ex: $x \cancel{+} - y + 2z = 3$
$x + y = 1$

$$J = \begin{pmatrix} (g_1)'_x & (g_1)'_y & (g_1)'_z \\ (g_2)'_x & (g_2)'_y & (g_2)'_z \end{pmatrix}$$

(Jacobian matrix)

$g_1(x,y,z) = x - y + 2z$
$g_2(x,y,z) = x + y$

$$J = \begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 2 \\ 0 \end{matrix} \end{pmatrix}$$

NDCQ: rk $J = 2$ at all adm. pts

In this case: $M_{12,12} = 1 + 1 = 2 \neq 0$
$\Rightarrow \text{rk } J = 2$

NDCQ is satisfied at all pts

No candidate pts where NDCQ fails

Ex: $\underbrace{2x^2 + y^2 + 3z^2 = 24}$ $\quad J = \begin{pmatrix} 4x & 2y & 6z \end{pmatrix}$

NDCQ: rk $J = 1$
NDCQ fails: rk $J < 1$ $\longrightarrow$ $4x = 0 \quad 2y = 0 \quad 6z = 0$
$\Updownarrow$ $\qquad x = 0 \quad y = 0 \quad z = 0$
rk $J = 0$ $\qquad (x,y,z) = (0,0,0)$ NDCQ fails
not admissible

NDCQ holds for all adm. pts

Ex: $x^2 - y^3 = 0$ $\qquad J = (2x \;\; 3y^2)$

NDCQ: $\text{rk } J = 1$

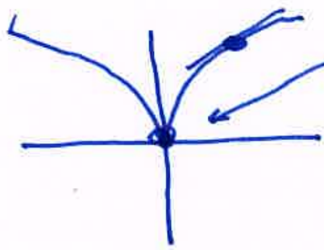
NDCQ fails: $2x = 3y^2 = 0$

$\Rightarrow (0,0)$

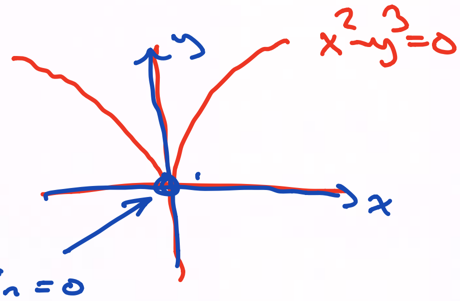
Adm pts where NDCQ fails: $(x,y) = (0,0)$

$y^3 = x^2$

$y = \sqrt[3]{x^2}$



$$\boxed{\min \; f(x,y) = y \quad \text{when} \quad x^2 - y^3 = 0}$$



$f_{min} = 0$ at $(0,0)$

a) Ordinary candidate pts:

$L = y - \lambda(x^2 - y^3)$

FOC $\begin{cases} L'_x = -\lambda \cdot 2x = 0 \\ L'_y = 1 - \lambda \cdot (-3y^2) = 0 \end{cases}$

C $\qquad x^2 - y^3 = 0$

$\lambda = 0 \quad$ or $\quad x = 0$

$\lambda = 0$	$x = 0$
$1 = 0$ not possible	$0 = y^3$ $\;y = 0$; $1 - \lambda \cdot 0 = 0$ not possible

no ord. cand. pts.

b) Adm pts where NDCQ fails

$J = (2x \;\; -3y^2)$

$\text{rk } J < 1$: $(x,y) = (0,0)$

NDCQ in the Lagrange case:

$$\boxed{\begin{aligned} g_1(\underline{x}) &= a_1 \\ &\vdots \\ g_m(\underline{x}) &= a_m \end{aligned}}$$

NDCQ: $\text{rk}\, J = \text{rk}\begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \cdots & \partial g_1/\partial x_n \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 & \cdots & \\ \vdots & & & \\ \partial g_m/\partial x_1 & \partial g_m/\partial x_2 & \cdots & \end{pmatrix}$ is maximal

$m \times n$ matrix

Usually, $m < n$, and then: NDCQ: $\text{rk}\, J = m$

NDCQ fails: $\text{rk}\, J < m \iff$ all m-minors are zero

check if the pts are adm.

NDCQ in the Kuhn-Tucker case:

$$\boxed{\begin{aligned} g_1(\underline{x}) &\le a_1 \\ &\vdots \\ g_m(\underline{x}) &\le a_m \end{aligned}}$$

For a pt $\underline{x}^*$ that is adm, look at those constraints that are binding (holds with $=$)

NDCQ: $\text{rk}\, (J')$ is maximal

J': Jacobian made from those constraints that are binding

Ex: $\boxed{\begin{aligned} x-y+2z &\ge 3 \\ x+y &\ge 3 \end{aligned}}$

No adm pts. where NDCQ fails. adm. under the case we are looking at

i) $x-y+2z=3$
$x+y=3$

$\text{rk}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = 2$ ok

$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \ne 0$

ii) $x-y+2z=3$
$x+y>3$

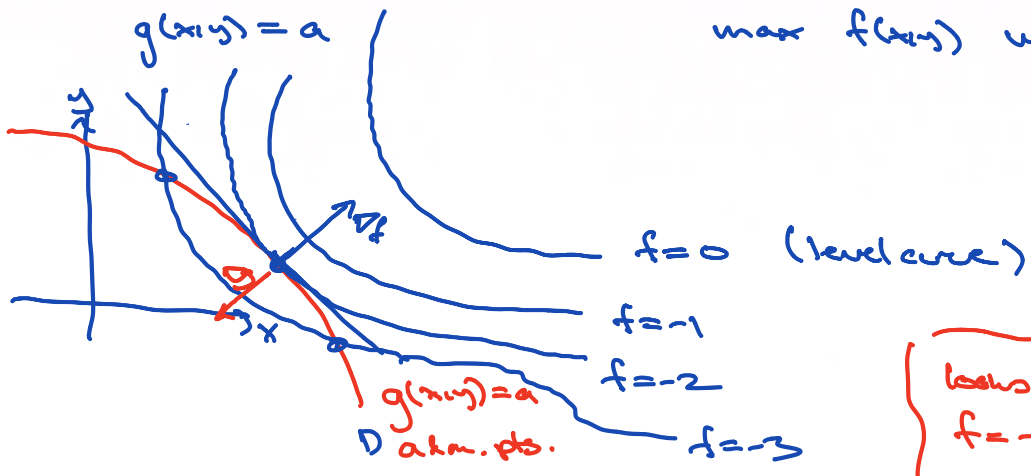
$\text{rk}\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} = 1$ ok

iii) $x-y+2z>3$
$x+y=3$

$\text{rk}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} = 1$ ok

iv) $x-y+2z>3$, $x+y>3$? always ok

Why? max $f(x,y)$ when $g(x,y)=a$



looks like $f=-2$ is the max value of f when $g(x,y)=a$.

Candidate pts: the constraint set D and the level curve of f meet at a tangent

$$\Updownarrow$$

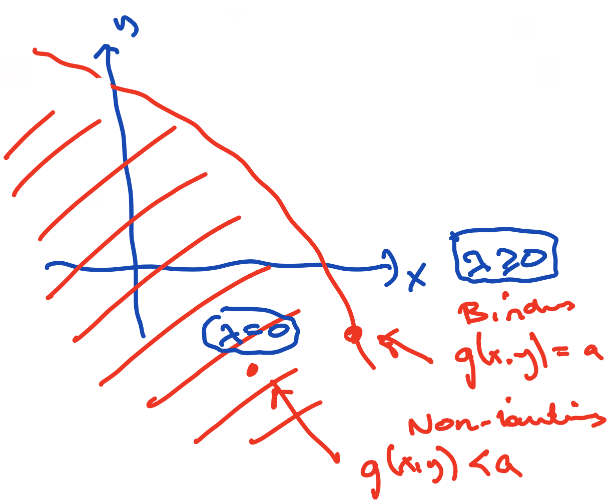
$$\nabla f = \lambda \cdot \nabla g$$

$$\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} \qquad \nabla g = \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$

$$\text{FOC} \begin{cases} L'_x = f'_x - \lambda g'_x = 0 \\ L'_y = f'_y - \lambda g'_y = 0 \end{cases} \qquad \begin{cases} f'_x = \lambda \cdot g'_x \\ f'_y = \lambda \cdot g'_y \end{cases} \Leftrightarrow \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \lambda \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$

NDCQ: $\operatorname{rk}\begin{pmatrix} g'_x & g'_y \end{pmatrix} = 1$ NDCQ fails: $\nabla g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Kuhn-Tucker case: $g(x,y) \le a$



interior pts: $\lambda = 0$ (ordinary stationary pts, NDCQ is no longer an issue)