Plan

1 Constrained optimization: Necessary conditions
2 Lagrange problems
3 Kuhn-Tucker problems

Note:

1) Plenary Session 2
Mon 17-20. (L4-6)

2) Mock midterm exam
next Fri at 15-16.

Review:

- unconstrained optimization

max/min $f(x_1, x_2, \dots, x_n)$

i) find stat. pts of f
ii) classify the loc. (sec. der. test)
iii) concl. about max/min

- convex/concave fn.

f convex $\Leftrightarrow$ $H(f)(\underline{x})$ pos. semidef. for all $\underline{x}$
f concave $\Leftrightarrow$ $H(f)(\underline{x})$ neg. — " — for all $\underline{x}$

f convex: any stat. pt is global min $\cup$
f concave any stat. pt is global max $\cap$

① Constrained optimization

max/min $f(x_1, x_2, \dots, x_n)$ with constraints { one or more equations or inequalities

Ex: min $f(x,y) = xy$ when $x^2 + y^2 = 10$

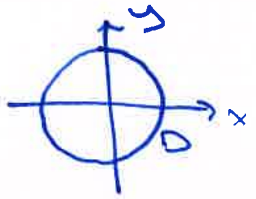
max $f(x,y,z,w) = xw - yz$ when $\begin{cases} x+y+z+w \le 16 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \\ w \ge 0 \end{cases}$

- objective function: $f(\underline{x}) = f(x_1, \dots, x_n)$
- admissible pts: $D = \{ \underline{x} = (x_1, \dots, x_n) \mid$ all constraints are satisfied $\}$
the set of all pts in $\mathbb{R}^n$ such that all constraints are satisfied

D subset of $\mathbb{R}^n$
set of admissible pts

The set D of admissible pts:

Ex: $x^2+y^2=10$

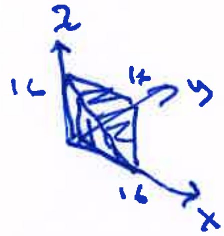


D: circle with $r=\sqrt{10}$, center $(0,0)$

min $f=xy$ when $x^2+y^2=10$

$$x+y+z+w \leq 16$$
$$x \geq 0, y \geq 0, z \geq 0, w \geq 0$$

D in $\mathbb{R}^4$



bounded:

$0 \leq x \leq 16$
$0 \leq y \leq 16$
$0 \leq z \leq 16$
$0 \leq w \leq 16$

D: set of admissible pts.

Defn: D is compact if it is closed and bounded

D is closed: when all constraints are given by = (equations) or $\leq, \geq$ (closed inequalities)

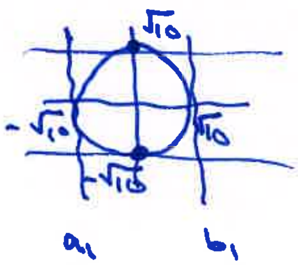
all boundary pts are in D

D is bounded: There exists bounds $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ (finite numbers) such that

$$a_1 \leq x_1 \leq b_1$$
$$a_2 \leq x_2 \leq b_2$$
$$\vdots$$
$$a_n \leq x_n \leq b_n$$

for all pts $(x_1, x_2, \ldots, x_n)$ in D.

Ex: $x^2+y^2=10$



$b_2=\sqrt{10}$

$a_2=-\sqrt{10}$

$a_1 = -\sqrt{10}$ $\quad b_1 = \sqrt{10}$

Extreme value theorem (EVT)

A continuous fn $f(x_1,\dots,x_n)$ on a compact set D in $\mathbb{R}^n$ has a (global) max and min

② Lagrange problems

max/min $f(x_1,\dots,x_n)$ when $\begin{cases} g_1(x_1,\dots,x_n) = a_1 \\ g_2(x_1,\dots,x_n) = a_2 \\ \vdots \\ g_m(x_1,\dots,x_n) = a_m \end{cases}$

n variables

m constraints (equalities)

$a_1,\dots,a_m$: constants

$g_1,\dots,g_m$: any function

Ex: min $f(x,y) = xy$ when $\underbrace{x^2+y^2}_{g(x,y)} = \underbrace{10}_{a}$

$\mathcal{L} = \mathcal{L}(x,y;\lambda) = f(x,y) - \lambda\cdot g(x,y)$

$= xy - \lambda\cdot(x^2+y^2)$

λ: Lagrange multiplier

$$\boxed{\begin{aligned} \mathcal{L}'_x &= y - \lambda\cdot 2x = 0 \\ \mathcal{L}'_y &= x - \lambda\cdot 2y = 0 \\ & x^2+y^2 = 10 \end{aligned}}$$

} FOC = first order conditions

} C = constraint

FOC+C = Lagrange conditions

Points $(x,y;\lambda)$ satisfying FOC+C: candidate pts

Thm Necessary conditions for Lagrange problems

If $(x_1^*, x_2^*, \dots, x_n^*)$ is a solution to the Lagrange problem, and $(x_1^*, \dots, x_n^*)$ satisfies NDCQ, then there are Lagrange multipliers $\lambda_1, \dots, \lambda_m$ such that $(x_1^*, x_2^*, \dots, x_n^*; \lambda_1, \dots, \lambda_m)$ satisfy FOC + C.

Problems: i) We have to check NDCQ
ii) are necessary conditions

max/min $\Longrightarrow$ candidate pt. (satisfying FOC+C) or pt. where NDCQ fails

Ex: min $f = xy$ when $g(x,y) = x^2 + y^2 = 10$

$\mathcal{L} = xy - \lambda(x^2 + y^2)$

$\mathcal{L}'_x = y - \lambda \cdot 2x = 0$
$\mathcal{L}'_y = x - \lambda \cdot 2y = 0$
$x^2 + y^2 = 10$

$y = 2\lambda x$
$x - 2\lambda y = x - 2\lambda(2\lambda x) = 0$
$x - 4\lambda^2 x = 0$
$x \cdot (1 - 4\lambda^2) = 0$
$x = 0$ or $\lambda^2 = 1/4$
$\lambda = \pm 1/2$

a)
$x = 0$
$y = 0$
$x^2 + y^2 = 0 \neq 10$
no solutions

b)
$\lambda = 1/2$
$y = x$
$2x^2 = 10$
$x^2 = 5$
$x = \pm\sqrt{5}$
$f = 5$
$(x, y; \lambda) = (\sqrt{5}, \sqrt{5}; \frac{1}{2})$, $(-\sqrt{5}, -\sqrt{5}; 1/2)$
$f = 5$

c)
$\lambda = -1/2$
$y = -x$
$2x^2 = 10$
$x^2 = 5$
$x = \pm\sqrt{5}$
$(x, y; \lambda) = (\sqrt{5}, -\sqrt{5}; -1/2)$
$(-\sqrt{5}, \sqrt{5}; -1/2)$
$f = -5$

Conclusion:

Candidate pts: (FOC+C) $(\sqrt{5},\sqrt{5};\,1/2)$, $(-\sqrt{5},-\sqrt{5};\,1/2)$, $(\sqrt{5},-\sqrt{5};\,-1/2)$, $(-\sqrt{5},\sqrt{5};\,-1/2)$

$f=5$ $\quad f=5$ $\quad \underbrace{f=-5 \quad f=-5}_{\text{best candidates for min}}$

EVT: $x^2+y^2=10$ compact $\Longrightarrow$ there is a min

$\Downarrow$ no pts where NDCQ fails $\leftarrow$ Assume this

$f_{min}=-5$ is minimum value at $(\sqrt{5},-\sqrt{5})$, $(-\sqrt{5},\sqrt{5})$ with $\lambda=-1/2$

Ex: max/min $f(x,y,z,w)=xw-yz$ when $\begin{cases} \underbrace{x^2+y^2}_{g_1}=16 \\ \underbrace{4z^2+9w^2}_{g_2}=36 \end{cases}$

$$\mathcal{L} = f(x,y,z,w) - \lambda_1\cdot g_1(x,y,z,w) - \lambda_2\cdot g_2(x,y,z,w)$$
$$= xw - yz - \lambda_1(x^2+y^2) - \lambda_2(4z^2+9w^2)$$

$$\text{FOC}\left\{\begin{array}{l} \mathcal{L}'_x = w - 2\lambda_1 x = 0 \\ \mathcal{L}'_y = -z - 2\lambda_1 y = 0 \\ \mathcal{L}'_z = -y - 8\lambda_2 z = 0 \\ \mathcal{L}'_w = x - 18\lambda_2 w = 0 \end{array}\right. \qquad \text{C}\left\{\begin{array}{l} x^2+y^2 = 16 \\ 4z^2+9w^2 = 36 \end{array}\right.$$

y,z $\qquad x,w$

x, w:

$w - 2\lambda_1 x = 0$ $\quad w = 2\lambda_1 x$

$x - 18\lambda_2 w = 0$

$x - 18\lambda_2(2\lambda_1 x) = 0$

$x - 36\lambda_1\lambda_2 x = 0$

$x(1 - 36\lambda_1\lambda_2) = 0$

a) $\boxed{x=0,\ w=0}$ or b) $\boxed{\lambda_1\lambda_2 = 1/36,\ w = 2\lambda_1 x}$

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y, z:

$-z - 2\lambda_1 y = 0$ $\quad z = -2\lambda_1 y$

$-y - 8\lambda_2 z = 0$

$-y - 8\lambda_2(-2\lambda_1 y) = 0$

$-y + 16\lambda_1\lambda_2 y = 0$

$y(-1 + 16\lambda_1\lambda_2) = 0$

c) $\boxed{y=0,\ z=0}$ or d) $\boxed{\lambda_1\lambda_2 = 1/16,\ z = -2\lambda_1 y}$

$\boxed{x^2+y^2=16,\ 4z^2+9w^2=36}$

a) + c): $x = y = z = w = 0$ not possible

a) + d): $x = w = 0$, $\lambda_1\lambda_2 = 1/16$, $z = -2\lambda_1 y$

$y^2 = 16$ $\quad 4z^2 = 36$

$y = \pm 4$ $\quad z = \pm 3$

$-2\lambda_1 = \frac{z}{y} = \frac{\pm 3}{\pm 4}$

$-2\lambda_1 = \pm 3/4$

$\lambda_1 = \mp 3/8$

$\lambda_2 = \mp 1/6$

$f = xw - yz$

Candidates:

$f = -12$ $\quad (0, 4, 3, 0; -3/8, -1/6)$

$f = 12$ $\quad (0, 4, -3, 0; 3/8, 1/6)$

$f = 12$ $\quad (0, -4, 3, 0; 3/8, 1/6)$

$f = -12$ $\quad (0, -4, -3, 0; -3/8, -1/6)$

b) + c): Similar with a+d)

b) + d): no pts $\quad \lambda_1\lambda_2 = 1/36,\ \lambda_1\lambda_2 = 1/16$

Plan:

① Example from Part 1 (continued)

② Kuhn-Tucker problems

① Continued: b) + c) $y = z = 0$, $\lambda_1\lambda_2 = 1/36$, $w = 2\lambda_1 x$

$\boxed{\begin{array}{l} x^2+y^2=16 \\ 4z^2+9w^2=36 \end{array}}$

$x^2 = 16$ $\quad 9w^2 = 36$

$x = \pm 4$ $\quad w = \pm 2$

$2\lambda_1 = \frac{w}{x} = \frac{\pm 2}{\pm 4} = \pm 1/2$

$\lambda_1 = \pm 1/4$

$\lambda_2 = \pm 1/9$

$f = xw - yz$

Candidates:

$f=8$ $\quad (4,0,0,2;\ 1/4,\ 1/9)$

$f=-8$ $\quad (4,0,0,-2;\ -1/4,\ -1/9)$

$f=-8$ $\quad (-4,0,0,2;\ -1/4,\ -1/9)$

$f=8$ $\quad (-4,0,0,-2;\ 1/4,\ 1/9)$

Best cand:

max $\quad (0,4,-3,0), (0,-4,3,0)$ $\quad \lambda_1 = 3/8$ $\lambda_2 = 1/6$ $\quad f=12$

min $\quad (0,4,3,0), (0,-4,-3,0)$ $\quad \lambda_1 = -3/8$, $\lambda_2 = -1/6$ $\quad f=-12$

EVT: $x^2+y^2=16$, $4z^2+9w^2=36$ is compact $\quad$ (closed for Lagrange problem)

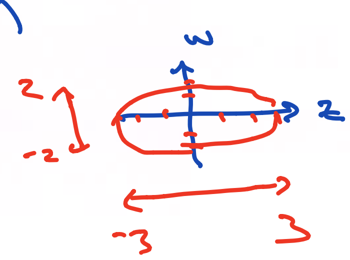
$\frac{z^2}{9} + \frac{w^2}{4} = 1$

bounded: $-4 \le x \le 4$

$-4 \le y \le 4$

$-3 \le z \le 3$

$-2 \le w \le 2$



There is a max and min

$\Downarrow$ $\boxed{\text{Assume NDCQ satisfied}}$

$f_{max} = 12$ $\quad f_{min} = -12$

② Kuhn-Tucker problems

Kuhn-Tucker problem: constrained optimization problem — all constraints are closed inequalities ($\leq$ or $\geq$)

$$\boxed{\max/\min\ f(x_1,\dots,x_n) \text{ when } \begin{cases} g_1(x_1,\dots,x_n) \leq / \geq a_1 \\ \vdots \\ g_m(x_1,\dots,x_n) \leq / \geq a_m \end{cases}}$$

KT-problem in standard form:

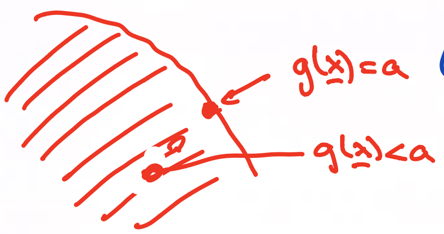
$$\max\ f(x_1,\dots,x_n) \text{ when } \begin{cases} g_1(x_1,\dots,x_n) \leq a_1 \\ \vdots \\ g_m(x_1,\dots,x_n) \leq a_m \end{cases}$$

Trick: $\min f(\underline{x}) = \max -f(\underline{x})$

$x^2+y^2 \geq 4 \quad | \cdot(-1)$

$-x^2-y^2 \leq -4$

Method:



$$\max\ f(x_1,\dots,x_n) \text{ when } \begin{cases} g_1(x_1,\dots,x_n) \leq a_1 \\ \vdots \\ g_m(x_1,\dots,x_n) \leq a_m \end{cases}$$

n variables $\qquad m$ inequalities

KT problem in std form

$$\mathcal{L} = f(x_1,\dots,x_n) - \lambda_1 g_1(x_1,\dots,x_n) - \lambda_2 \cdot g_2(x_1,\dots,x_n) - \dots - \lambda_m \cdot g_m(x_1,\dots,x_n)$$

FOC $\begin{cases} \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{cases}$ $\quad f'_{x_1} - \lambda \cdot g'_{x_1} = 0$

C: $\begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$

CSC: complementary slackness conditions $\quad \begin{array}{l} \lambda_1 \geq 0 \\ \lambda_2 \geq 0 \\ \vdots \\ \lambda_m \geq 0 \end{array}$

and

$$\boxed{\begin{array}{l} \lambda_1 = 0 \text{ if } g_1(\underline{x}) < a_1 \\ \lambda_2 = 0 \text{ if } g_2(\underline{x}) < a_2 \\ \vdots \\ \lambda_m = 0 \text{ if } g_m(\underline{x}) < a_m \end{array}} \iff \begin{cases} \lambda_1 (g_1(\underline{x}) - a_1) = 0 \\ \lambda_2 (g_2(\underline{x}) - a_2) = 0 \\ \vdots \\ \lambda_m (g_m(\underline{x}) - a_m) = 0 \end{cases}$$

Constraint i is binding if $g_i(\underline{x}) = a_i$

and non-binding if $g_i(\underline{x}) < a_i$

Kuhn–Tucker conditions: $\boxed{FOC + C + CSC}$

Pts $(x_1,\dots,x_n;\lambda_1,\dots,\lambda_m)$ that satisfy FOC + C + CSC are candidate pts.

Ex: max $f = xy$ when $x^2+y^2 \le 10$ $\boxed{\text{KT-problem std. form}}$

$L = xy - \lambda(x^2+y^2)$

FOC $\begin{cases} L'_x = y - \lambda\cdot 2x = 0 \\ L'_y = x - \lambda \cdot 2y = 0 \end{cases}$

C $\quad x^2+y^2 \le 10$

CSC $\begin{cases} \lambda \ge 0 \\ \lambda(x^2+y^2-10) = 0 \ ✓ \end{cases}$

KT conditions
FOC + C + CSC

Binding	Non-binding
$y - 2\lambda x = 0$	$y - 2\lambda x = 0$ ✓
$x - 2\lambda y = 0$	$x - 2\lambda y = 0$ ✓
$x^2+y^2 = 10$	$x^2+y^2 < 10$ ok.
$\lambda \ge 0$	$\lambda = 0$ ✓
$(x,y;\lambda) =$	$y = 0$
$(\sqrt5, \sqrt5; 1/2)$ $f=5$	$x = 0$
$(-\sqrt5, -\sqrt5; 1/2)$ $f=5$	$(x,y;\lambda) =$
~~$(\sqrt5, -\sqrt5; -1/2)$~~	$(0,0;0)$ $f=0$
~~$(-\sqrt5, \sqrt5; -1/2)$~~	

Best cand: $\boxed{(\sqrt5,\sqrt5),\ (-\sqrt5,-\sqrt5) \quad \lambda = 1/2 \quad f=5 \quad f=5}$

EVT:
$x^2+y^2 \le 10$

closed $(\le)$
bounded ✓ $\begin{cases} -\sqrt{10} \le x \le \sqrt{10} \\ -\sqrt{10} \le y \le \sqrt{10} \end{cases}$
compact $\Rightarrow$ there is a max

$\Downarrow$ $\boxed{\text{Assume NDCQ holds}}$

$f_{max} = 5$ at $(x,y) = (\sqrt5, \sqrt5), (-\sqrt5, -\sqrt5)$ with $\lambda = 1/2$

Thm (Necessary conditions for KT problems)

If $(x_1^*, \ldots, x_n^*)$ solves the KT problem, and if NDCQ is satisfied, then there exists $\lambda_1, \ldots, \lambda_m$ such that $(x_1^*, \ldots, x_n^*; \lambda_1, \ldots, \lambda_m)$ satisfy FOC + C + CSC.

$$(x_1^*, \ldots, x_n^*) \text{ max } \Rightarrow \begin{cases} (x_1^*, \ldots, x_n^*; \lambda_1, \ldots, \lambda_m) \text{ satisfy FOC + C + CSC} \\ \text{or} \\ (x_1^*, \ldots, x_n^*) \text{ fails NDCQ} \end{cases}$$

Reminder: Plenary Session 2
Monday 17-20

tell me if there are pb's from Lecture 4-6 you want me to go through