Plan

1. Eigenvalues and eigenvectors
2. Diagonalization
3. Applications: Markov chains

Plenary Session I: Monday A1-040 17-20
- no TA session
- problems Lecture 1-4

Review:
- matrix multiplication, determinants and minors
- for an $n \times n$ matrix A: (square) $\boxed{rk(A) = n \Leftrightarrow |A| \neq 0}$

$(v_1 | v_2 | \dots | v_n)$ $\{v_1, \dots, v_n\}$ lin. indep. A invertible

$A\underline{x} = \underline{b}$ has unique sol.

- for an $m \times n$ matrix A: (general) $rk\, A$ = maximal order of a non-zero minor

$\boxed{rk\, A < r \Leftrightarrow \text{all } r\text{-minors are zero}}$

Ex:

$$\begin{aligned} x + y + 2z &= 8 \\ x - y + 4z &= 6 \\ \cancel{3x + y + 8z = 22} \end{aligned}$$

no pivot here, consist.

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 4 \\ 3 & 1 & 8 \end{vmatrix} = 0$$

$$M_{12,12} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

$M_{12,12} \neq 0$ maximal non-zero minor in A

$$x + y = 8 - 2z$$
$$x - y = 6 - 4z$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8-2z \\ 6-4z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 8-2z \\ 6-4z \end{pmatrix}$$

$$= \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 8-2z \\ 6-4z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\begin{pmatrix} 7-3z \\ 1+z \end{pmatrix}}$$

$$3 \cdot (7-3z) + 1 \cdot (1+z) + 8z$$

ok $= 22$

Concl: x, y basic, z free. Solution $(x,y,z) = (7-3z, 1+z, z)$

1 Eigenvalues and eigenvectors

A
$n \times n$
matrix

Defn: A number λ is called an eigenvalue if $\{A \cdot \underline{v} = \lambda \cdot \underline{v}\}$ has non-trivial solutions. In that case, we call the vectors in $E_\lambda = \{$all solutions $\underline{v}$ of $A\underline{v} = \lambda \underline{v}\}$ are called eigenvectors of A with eigenvalue λ, and E_λ is called the eigenspace.

Remark:

$$A \cdot \underline{v} = \lambda \cdot \underline{v}$$

$$\lambda \cdot \underline{v} = \lambda \cdot I \cdot \underline{v} = (\lambda I)\underline{v}$$

$$A\underline{v} - \lambda \underline{v} = 0 \qquad (A - \lambda I)\underline{v} = \underline{0}$$

$$\boxed{A\underline{v} = \lambda \underline{v} \iff (A - \lambda I)\underline{v} = \underline{0}}$$

homogeneous $n \times n$ linear system with parameter λ

Fact:

1) λ eigenvalue $\iff |A - \lambda I| = 0$ characteristic eqn.

2) When λ is an eigenvalue: $E_\lambda = \text{Null}(A - \lambda I)$ eigenvectors with eigenvalue λ

Ex: $A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

i) Eigenvalues: $\begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$$(4-\lambda)(1-\lambda) - 4 = 0$$

$$\lambda^2 - 5\lambda + 4 - 4 = 0$$

$$\lambda^2 - 5\lambda = 0 \quad \rightarrow \quad \lambda_1 = 5, \ \lambda_2 = 0$$

$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

$\lambda_1 = 5$, $\lambda_2 = 0$

2) Eigenvectors:

$\lambda = 5$: $E_5 = \text{Null}(A - 5I)$

$= \text{Null}\begin{pmatrix} -1 & 2 \\ \cancel{2} & \cancel{-4} \end{pmatrix}$

$-1x + 2y = 0$ $\quad x = 2y$, y free

$\cancel{2x - 4y = 0}$

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix}$

$= y \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$: base for E_5

$\lambda = 0$: $E_0 = \text{Null}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

$4x + 2y = 0$ $\quad y = -2x$, x free

$\cancel{2x + y = 0}$

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -2x \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\underline{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$: base for E_0

Alt:

$x = -\frac{y}{2}$, y free

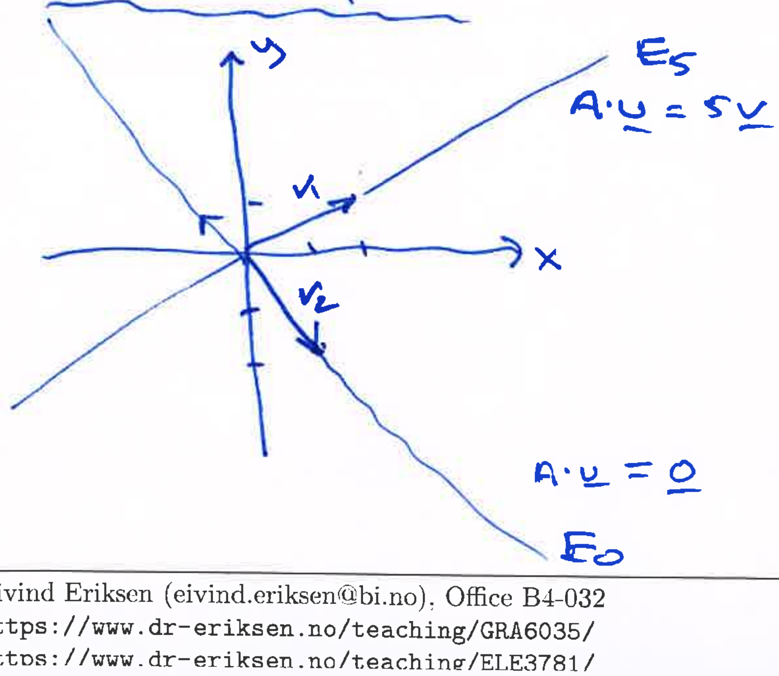
$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y/2 \\ y \end{pmatrix}$

$= y \cdot \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$\underline{v}_2 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ base for E_0

$\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ lin. independent eigenvectors of

Geometric picture:



$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

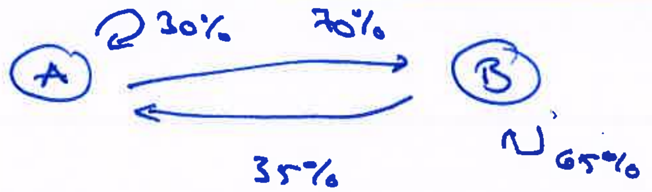
$A \cdot \underline{v} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$= \begin{pmatrix} 4x + 2y \\ 2x + y \end{pmatrix}$

$A \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} = 5 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$A \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Example:



Markov chain

Company A and B share a market

$$\underline{v_0} = \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix}$$ State vector

$$A = \begin{array}{cc} A\to A & B\to A \\ \downarrow & \downarrow \\ \left(\begin{matrix} 0.30 \\ 0.70 \end{matrix}\right. & \left.\begin{matrix} 0.35 \\ 0.65 \end{matrix}\right) \\ \uparrow & \uparrow \\ A\to B & B\to B \end{array}$$

$$\underline{v_1} = A \cdot \underline{v_0} = \begin{pmatrix} 0.30 & 0.35 \\ 0.70 & 0.65 \end{pmatrix} \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix}$$

after 1 time period

$$= \begin{pmatrix} 0.30 \cdot 0.88 + 0.35 \cdot 0.12 \\ 0.70 \cdot 0.88 + 0.65 \cdot 0.12 \end{pmatrix}$$

$$= \begin{pmatrix} 0.306 \\ 0.694 \end{pmatrix}$$

$$\underline{v_n} = A^n \cdot \underline{v_0}$$

after n time periods

Finding eigenvalues: Solve the characteristic equation.

$|A-\lambda I|=0$

characteristic eqn.

i) The case $n=2$:

$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=0$

$(a-\lambda)(d-\lambda)-bc=0$

$\lambda^2-a\lambda-d\lambda+ad-bc=0$

$\lambda^2-(a+d)\lambda+(ad-bc)=0$

$$\boxed{\lambda^2-tr(A)\lambda+det(A)=0}$$

$tr(A)$ = sum of diagonal entries in A

ii) The general case:

$A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}=0$

$(-1)^n\cdot\lambda^n+\ldots=0$

Facts:

i) The characteristic eqn. is a n'th order eqn.

ii) If A is symmetric, it has n solutions $\lambda_1,\lambda_2,\ldots,\lambda_n$.

If A is not symmetric, there could be fewer than n solutions.

iii) If there are n solutions $\lambda_1,\ldots,\lambda_n$: $\begin{cases} \lambda_1\cdot\lambda_2\cdot\ldots\cdot\lambda_n=det(A) \\ \lambda_1+\lambda_2+\ldots+\lambda_n=tr(A) \end{cases}$

$n=3$

Ex:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$+(2-\lambda)\cdot\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\cdot(\lambda^2-6\lambda+8) = 0$$

$\underline{\lambda_1 = 2}$ or $\lambda^2 - 6\lambda + 8 = 0$

$\underline{\lambda_2 = 2}, \ \underline{\lambda_3 = 4}$

$\lambda = 2$ has multiplicity 2

$$-(\lambda-2)(\lambda-2)(\lambda-4) = 0$$

$$-(\lambda-2)^2\cdot(\lambda-4) = 0$$

$$\boxed{\begin{array}{ll} \lambda=2 & \text{mult.}=2 \\ \lambda=4 & \text{mult.}=1 \end{array}}$$

2) Finding eigenvectors for λ: Solve linear system

$E_\lambda = \text{Null}(A-\lambda I)$: $(A-\lambda I)\,\underline{v} = \underline{0}$

solve the linear system $\to$ Gauss

Fact: i) $1 \leq \dim E_\lambda \leq \text{mult}(\lambda)$

ii) If A is symmetric, then $\dim E_\lambda = \text{mult}(\lambda)$.

$\dim E_\lambda$ = # free variables in the linear system

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② Diagonalizable matrices

A $n \times n$ matrix

Defn: A is diagonalizable if there is an invertible matrix P such that

$$P^{-1}AP = D$$

is diagonal,

Results:

A is diagonalizable if and only if the following holds:

i) there are n eigenvalues of A (counted with multiplicity)

ii) for each eigenvalue λ, we have

$$\dim E_\lambda = \text{mult}(\lambda)$$

If A is symmetric, then it is diagonalizable

In that case:

$$P = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n)$$

where $\{\underline{v}_1, \dots, \underline{v}_n\}$ are linearly independent eigenvectors

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Method:

- Find all eigenvalues of A. If there are n $(\lambda_1, \dots, \lambda_n)$ $\longrightarrow$ $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$
- For each λ, find base of E_λ, and join all these bases. If there are n vectors $\longrightarrow$ $P = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n)$

Plan:

1. More examples: Eigenvalues and eigenvectors
2. More on diagonalization and computation of A^m
3. Markov chain

1 Examples: Eigenvalues and eigenvectors

Ex: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ not symmetric

Eigenvalues:

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$\lambda^2 = -1$ no real solutions

not diagonalizable

Ex: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$(\lambda - 1)^2 = 0$ $\lambda_1 = \lambda_2 = 1$ (mult. 2)

Eigenvectors:

$\lambda = 1$: (mult = 2) $A - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$0 \cdot x + 1 \cdot y = 0$$
$$0 \cdot x + 0 \cdot y = 0$$

x free, $y = 0$

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\left\{ \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$: base of E_1

Diagonalization: 1) ok enough eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1$

2) not ok. $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ no $\underline{v}_2$

A not diagonalizable

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Ex: $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\cdot\left[(2-\lambda)^2-1\right] - 1\cdot\left((2-\lambda)-1\right) + 1\cdot\left(1-(2-\lambda)\right) = 0$$

$$(2-\lambda)\cdot\left((2-\lambda)^2-1\right) + \underbrace{\lambda-1 + \lambda-1} = 0$$

$$(2-\lambda)\cdot(\lambda^2-4\lambda+3) + 2(\lambda-1) = 0$$

$$(2-\lambda)(\lambda-3)(\lambda-1) + 2(\lambda-1) = 0$$

$$(\lambda-1)\cdot\left[(2-\lambda)(\lambda-3) + 2\right] = 0$$

$\lambda = 1$ $\quad -\lambda^2+5\lambda-4=0$

$\lambda^2-5\lambda+4=0$

$\lambda=1, \quad \lambda=4$

$\Big\}$ $\lambda_1=\lambda_2=1$ (mult 2)

$\lambda_3=4$ (mult 1)

Eigenvectors:

$\lambda=1$: $\begin{pmatrix} 1 & 1 & 1 \\ \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{1} & \cancel{1} \end{pmatrix}$ $\quad x+y+z=0$

y, z free

$x = -y-z$

$\Big\}$ $\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Base: $\left\{ \underline{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ \underline{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\lambda=4$: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{matrix} \\ ]-1 \\ ]2 \end{matrix}$

$\to \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}]1 \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$x+y-2z=0$ $\quad x = -y+2z = z$

$-3y+3z=0$ $\quad y = z$

$\Big\}$ $\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Base: $\left\{ \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Conclusion: A is diagonalizable ✓ $\left\{ \begin{array}{l} 1)\ \lambda_1=\lambda_2=1,\ \lambda_3=4 \\ 2)\ \underline{v}_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}, \underline{v}_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}, \underline{v}_3=\begin{pmatrix}1\\1\\1\end{pmatrix} \end{array}\right.$

$$\underbrace{P=\begin{pmatrix}-1&-1&1\\1&0&1\\0&1&1\end{pmatrix} \quad D=\begin{pmatrix}1&0&0\\0&1&0\\0&0&4\end{pmatrix}}$$

$$P^{-1}AP=D$$

② More on diagonalization:

Requirements: $\left\{ \begin{array}{l} \text{i) } n \text{ eigenvalues } \lambda_1,\dots,\lambda_n \\ \text{ii) } n \text{ linearly independent eigenvectors } \underline{v}_1,\dots,\underline{v}_n \end{array}\right.$

Note: a) If A is symmetric, then it is diagonalizable
b) If A has n distinct (mult. 1) eigenvalues ~~then~~ A is diagonalizable.

Explanation: $P=(\underline{v}_1|\underline{v}_2|\dots|\underline{v}_n)$: $D=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$

$$\begin{aligned} AP &= A\cdot(\underline{v}_1|\underline{v}_2|\dots|\underline{v}_n) \\ &= (A\underline{v}_1|A\underline{v}_2|\dots|A\underline{v}_n) \\ &= (\lambda_1\underline{v}_1|\lambda_2\underline{v}_2|\dots|\lambda_n\underline{v}_n) \\ &= (\underline{v}_1|\underline{v}_2|\dots|\underline{v}_n)\cdot\begin{pmatrix}\lambda_1 & & 0\\ & \lambda_2 \ddots & \\ 0 & & \lambda_n\end{pmatrix} \\ &= P\cdot D \end{aligned}$$

$AP=PD \quad |P^{-1}\cdot$

$\boxed{P^{-1}AP=D}$

$|\cdot P^{-1}$

$\boxed{A=PDP^{-1}}$

Computation of A^m: – we can use a diagonalization of A to compute this

$P^{-1}AP=D \Rightarrow A=\underline{PDP^{-1}}$

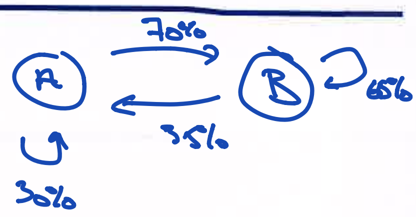
$$A^m=(PD\cancel{P^{-1}})(\cancel{PD}\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\dots(\cancel{P}DP^{-1})$$

$$\boxed{A^m=PD^mP^{-1}}$$

Ex: $P=\begin{pmatrix}-1&-1&1\\1&0&1\\0&1&1\end{pmatrix}$ $\quad D=\begin{pmatrix}1&0&0\\0&1&0\\0&0&4\end{pmatrix}$

$P^{-1}=\dots$ $\quad D^m=\begin{pmatrix}1^m&0&0\\0&1^m&0\\0&0&4^m\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&4^m\end{pmatrix}$

③ Markov chains: An example



Transition matrix:

$$A=\begin{pmatrix}0.30 & 0.35\\ 0.70 & 0.65\end{pmatrix} \qquad u_0=\begin{pmatrix}0.88\\ 0.12\end{pmatrix}$$

Initial state:

$$\underline{v_0}=\begin{pmatrix}0.88\\ 0.12\end{pmatrix}$$

$$\boxed{\text{Equation: } \underline{v_{t+1}} = A\cdot \underline{v_t}}$$

$$\underline{v_1} = A\cdot \underline{v_0}$$

$$\underline{v_2} = A\cdot \underline{v_1} = A^2\cdot \underline{v_0}$$

$$\vdots$$

$$\boxed{\underline{v_m} = A^m\cdot \underline{v_0}}$$

long term market shares when $m \gg 0$

Eigenvalues of A:

$$\begin{vmatrix}0.30-\lambda & 0.35\\ 0.70 & 0.65-\lambda\end{vmatrix}=0$$

$$\lambda^2-0.95\lambda+(0.30\cdot 0.65-0.70\cdot 0.35)=0$$

$$\underline{\lambda_1=1}, \quad \lambda_2=\underline{-0.05}$$

Eigenvectors:

$$\lambda=1: \quad \begin{pmatrix}-0.70 & 0.35\\ \cancel{0.70} & \cancel{-0.35}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$-0.70x+0.35y=0 \rightarrow \left.\begin{array}{l} y=2x\\ x \text{ free}\end{array}\right\} \begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ 2x\end{pmatrix} = \underline{\underline{x\cdot\begin{pmatrix}1\\ 2\end{pmatrix}}}$$

Equilibrium state: $\underline{\underline{\underline{v}=\begin{pmatrix}1/3\\ 2/3\end{pmatrix}}}$ with $x=1/3$

$$\underline{\lambda=-0.05}: \quad \begin{pmatrix}0.35 & 0.35\\ \cancel{0.70} & \cancel{0.70}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \qquad \begin{array}{l} x=-y\\ y \text{ free}\end{array} \qquad \begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-y\\ y\end{pmatrix} = \underline{\underline{y\cdot\begin{pmatrix}-1\\ 1\end{pmatrix}}}$$

Long term:

$$A^m = P\cdot D^m\cdot P^{-1} = \overset{P}{\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}}\cdot\begin{pmatrix}1 & 0\\ 0 & -0.05\end{pmatrix}^m\cdot \overset{P^{-1}}{\frac{1}{3}\begin{pmatrix}1 & 1\\ -2 & 1\end{pmatrix}}$$

$$=\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \underset{\downarrow\, 0}{(-0.05)^m}\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}1 & 1\\ -2 & 1\end{pmatrix}$$

$$\xrightarrow{\boxed{m\to\infty}} \begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix}\cdot\frac{1}{3}\cdot\begin{pmatrix}1 & 1\\ -2 & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix} = \underline{\underline{\begin{pmatrix}1/3 & 1/3\\ 2/3 & 2/3\end{pmatrix}}}$$

$\underline{v_m} \to \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix} \cdot \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix} = 0.88 \cdot \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} + 0.12 \cdot \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}}}$

$\uparrow$

$\underline{v_0}$

<u>Long term market shares:</u>

(independent of the initial state)

$\boxed{\begin{matrix} A: & 1/3 \\ B: & 2/3 \end{matrix}}$