Plan

1 Matrices and matrix algebra
2 Determinants and minors
3 Minors, rank and linear systems

Review:

- span and linear independence of vectors
- bases and dimension of vector spaces

$A = (\underline{v_1}|\underline{v_2}|\dots|\underline{v_n})$
$m\times n$ matrix
vectors in $\mathbb{R}^m$

1) $\mathrm{Col}(A) = \mathrm{span}(\underline{v_1},\dots,\underline{v_n})$ vector space in $\mathbb{R}^m$
column space

Base: vectors corresp. to pivot positions

Dim: $\dim \mathrm{Col}(A) = \mathrm{rk}(A)$

2) $\mathrm{Null}(A) = \{$all solutions $\underline{x} = (x_1,\dots,x_n)$ in the linear system with augmented matrix $(A|\underline{0})\}$
nullspace

vector space in $\mathbb{R}^n$

Base: solve the linear system $\rightarrow$

$\{t_1,\dots,t_k\}$ free variables $\longrightarrow$ $t_1\underline{w_1} + t_2\underline{w_2} + \dots + t_k\cdot\underline{w_k}$

Base: $\{\underline{w_1},\underline{w_2},\dots,\underline{w_k}\}$

Dim: $\dim \mathrm{Null}(A) = n - \mathrm{rk}(A)$
(# free variables)

① Matrices and matrix algebra

An $m \times n$ matrix
$$A = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{n} \Big\} m$$

a_{ij}: entry in pos. (i,j) in the matrix A

(row i, col j)

Matrix operations:

i) **Addition / subtraction:** $A \pm B$

Ex: $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 3 \\ 0 & 2 & -1 \end{pmatrix}$

ii) **Scalar multiplication:** $r \cdot A$

Ex: $2 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$

iii) **Vectors as column vectors**

$\underline{v} = (1, 1, -1) \iff \underline{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ column vector = matrix with one column

iv) **Transpose:** A^T

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}$

Defn: A is symmetric if $A^T = A$.

v) Special matrices:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ identity matrix

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ diagonal matrix

$U = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix}$ upper triangular matrix

Note: a square echelon form is upper triangular

vi) Matrix multiplication: $\underset{m\times n}{A} \cdot \underset{n\times p}{B} \rightsquigarrow \underset{m\times p}{AB}$

i) Defined if # cols in A = # rows in B

ii) Result has the same number of rows as A and cols as B

Ex: $\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}} \cdot \underset{2\times 3}{\begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 6 & 1 \\ 3 & 12 & 3 \end{pmatrix}$

Properties:

i) $AB \neq BA$

ii) $A\cdot(B+C) = AB + AC$

iii) $A\cdot(BC) = (AB)\cdot C$

iv) $A \cdot I = A$ and $I \cdot A = A$ (multiplicative identity)

vii) Inverses: A^{-1}

Defn: A^{-1} is a matrix such that $\begin{cases} A\cdot A^{-1} = I \\ A^{-1}\cdot A = I \end{cases}$

i) A^{-1} does not always exist : A is called invertible if A^{-1} exists

ii) If A^{-1} exists, it is unique

<u>Application:</u> Matrix mult. and linear systems

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

coeff. matrix

$$\Updownarrow$$

$$\boxed{A \cdot \underline{x} = \underline{b}}$$

matrix form of the linear system

<u>Assume that A is invertible:</u>

$A \cdot \underline{x} = \underline{b} \quad | \; A^{-1}\cdot$

$A^{-1}(A\underline{x}) = A^{-1}\underline{b}$

$I \cdot \underline{x} = A^{-1}\underline{b}$

$\underline{x} = A^{-1}\underline{b}$ one solution

② <u>Determinants and minors</u>

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \rightsquigarrow \det(A) = |A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$n \times n$ matrix — determinant of A.

i) <u>Computation</u> by cofactor expansion.

<u>n=2:</u> $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $|A| = ad - bc$ <u>n=1:</u> $A = (a)$: $|A| = a$

<u>n>2:</u> Cofactor expansion

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \quad C = \begin{pmatrix} 6 & -5 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$|A| = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$

$= 1 \cdot (+) \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 1(-1) \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1(+1) \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$= 1(6) - 1(5) + 1(1) = \underline{\underline{2}}$

$C_{ij} = (-1)^{i+j} \cdot M_{ij}$, where M_{ij} = determinant of the submatrix you get when you delete row i, col j from A

Fact: i) You can do cofactor expansion along any row or column, and you get same result, $|A|$.

ii) If A is upper triangular, then the determinant of A is the product of the diagonal entries.

Ex: $A = \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad |A| = 1 \cdot (+1) \cdot \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}$

$= 1 \cdot (1 \cdot 2 - \cancel{4 \cdot 0})$

$= 1 \cdot 1 \cdot 2 = \underline{\underline{2}}$

Computation using Gaussian elimination:

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{matrix} -1 \\ \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} -2 \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = E$

$|A| = |E| = \underline{\underline{2}}$ $\qquad |E| = 1 \cdot 1 \cdot 2$

Fact: $A \to B$ elementary row op.

i) If you switch rows, then $|B| = -|A|$

ii) If you mult. a row with $c \neq 0$, then $|B| = c \cdot |A|$

iii) If you add a multiple of one row to another row, then $|B| = |A|$.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 0 & 4 \end{pmatrix} \begin{matrix} -2 \\ \\ -3 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -3 & 1 \end{pmatrix} -1 \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} = E$

$|A| = 0$ $\qquad |E| = 1 \cdot (-3) \cdot 0 = 0$

i) $\boxed{|A| = 0 \iff \text{rk}(A) < n}$ Important fact!

$A \to \cdots \to E$ echelon form $\qquad |A| = 0 \iff |E| = 0$

$$E = \begin{vmatrix} e_{11} & * & \cdots & * \\ 0 & e_{22} & \cdot & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e_{nn} \end{vmatrix}$$

$|E| = e_{11} \cdot e_{22} \cdot \ldots \cdot e_{nn} = 0$

$\Updownarrow$

at least one $e_{ii} = 0$

$\Updownarrow$

at least one column lacks a pivot

ii) If A has two equal rows, $|A| = 0$

If A has a row of zeros, $|A| = 0$

If A has a row that is a linear comb. of the other rows, $|A| = 0$.

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 0$ (-1, -2)

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{vmatrix} \; (-1) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{vmatrix}$$

iii) $\boxed{|A \cdot B| = |A| \cdot |B|} \qquad \boxed{|A^T| = |A|}$ ← This means that the same comment for rows in (ii) also applies to columns:

If a column is zero, then $|A| = 0$

If two columns are equal, " $|A| = 0$

If a column is a linear comb. of the other columns, " $|A| = 0$

Summary of results:

Let A be an $n \times n$ matrix, and let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ be the column vectors of A. Then the following statements are equivalent:

i) $\{\underline{v}_1, \ldots, \underline{v}_n\}$ are linearly independent vectors

ii) $rk(A) = n$

iii) $|A| \neq 0$

iv) A is invertible

v) $A\underline{x} = \underline{b}$ has a unique solution for any vectors $\underline{b}$

Note: $|A| \neq 0 \iff A^{-1}$ exists

In fact: $|A| \neq 0 \Rightarrow A^{-1} = \frac{1}{|A|} \cdot \underbrace{\begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}^T}_{adj(A)}$

$|A| = 0 \Rightarrow A^{-1}$ does not exist

Minors:

Defn: An <u>r-minor</u> of A is the determinant of an $r \times r$-submatrix of A

A $m \times n$ matrix

Ex: $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & 2 \end{pmatrix}$

2-minors: = maximal minors

$M_{12,12} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1+2 = \underline{3}$

$M_{12,23} = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 4-4 = \underline{0}$

$M_{12,13} = \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = 2+4 = \underline{6}$

we choose two rows (= all rows, "12") and two of the three columns (= "12", "23", "13")

<u>1-minors:</u> $M_{1,1} = 1$ $M_{1,2} = 2$ $M_{1,3} = 4$

$M_{2,1} = -1$ $M_{2,2} = 1$ $M_{2,3} = 2$

Result: $\text{rk}(A)$ is the size of a maximal non-zero minor in A

$$\text{rk}(A) = \max \{ r : \text{an } r\text{-minor is non-zero} \}$$

Ex: $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & 2 \end{pmatrix}$ has a non-zero 2-minor, for example $M_{12,12} = 3 \neq 0$, hence $\underline{\text{rk}\, A = 2}$

1. Powers of matrices
2. Minors, rank and linear systems
3. More examples of determinants and minors

1. Powers

$$\underset{n\times n}{A} \leadsto A^2 = A\cdot A$$

$$A^3 = A\cdot A\cdot A = A^2\cdot A$$

$$\vdots$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \underline{\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}}$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D^k = \begin{pmatrix} (-1)^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad k \geq 2$$

(2) Minors, rank and linear systems

Ex: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 4 \\ 3 & 0 & 4 & 6 \end{pmatrix}$ $\quad \underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$

3×4

$$A\underline{x} = \underline{b} \iff \begin{cases} x + y + z + w = 5 \\ 2x - y + 3z + 4w = 2 \\ 3x + 4z + 6w = -1 \end{cases}$$

i) Cannot compute $|A|$.

ii) Compute 3-minors:

$\boxed{\text{all 3-minor} = 0 \iff \text{rk } A < 3}$

$$M_{123,123} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 0 & 4 \end{vmatrix} = 0$$

$M_{12,12} = -3 \neq 0$

$$M_{123,124} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{vmatrix} = 3 \cdot 5 + 6 \cdot (-3) = -3 \neq 0$$

$\Rightarrow \text{rk}(A) = 3$ //

$\boxed{\text{infinitely many solutions, } z \text{ is free}}$

$$\begin{aligned} x + y + w &= 5 - z \\ 2x - y + 4w &= 2 - 3z \\ 3x + 6w &= -1 - 4z \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 5-z \\ 2-3z \\ -1-4z \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 5-z \\ 2-3z \\ -1-4z \end{pmatrix}$$

3 More examples

1) Find rank of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & 3 & 9 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & 3 & 9 \end{vmatrix} = 1\cdot(18-3a) - 1\cdot(9-a) + 1\cdot 1$$

3-minor

$$= 18 - 3a - 9 + a + 1$$

$$= \underline{10 - 2a}$$

$|A| = 0$
$10 - 2a = 0$
$\underline{a = 5}$

$\underline{a=5}$: $|A| = 0 \Rightarrow \text{rk}\, A \leq 2$

$M_{12,12} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0 \Rightarrow \text{rk}\, A = \underline{2}$

$\underline{a \neq 5}$: $|A| \neq 0 \Rightarrow \text{rk}\, \underline{A = 3}$

Conclusion: $\text{rk}(A) = \begin{cases} 3, & a \neq 5 \\ 2, & a = 5 \end{cases}$

2)

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} \begin{matrix} \leftarrow \\ \\ \\ \\ \end{matrix} \begin{matrix} -4 \\ \\ \\ -1 \end{matrix} = \begin{vmatrix} 0 & -4 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= +1 \cdot \begin{vmatrix} -4 & 0 & -1 & -1 \\ 2 & 0 & 1 & -1 \\ 0 & 6 & -2 & 0 \\ -2 & 2 & 0 & 0 \end{vmatrix} \; -3 = \begin{vmatrix} -4 & 0 & -1 & -1 \\ 2 & 0 & 1 & -1 \\ 6 & 0 & -2 & 0 \\ -2 & 2 & 0 & 0 \end{vmatrix}$$

$$= +2 \cdot \begin{vmatrix} -4 & -1 & -1 \\ 2 & +1 & -1 \\ 6 & -2 & 0 \end{vmatrix} = 2\left(6 \cdot 2 - (-2) \cdot 6\right)$$

$$= 2(12 + 12) = \underline{\underline{48}}$$