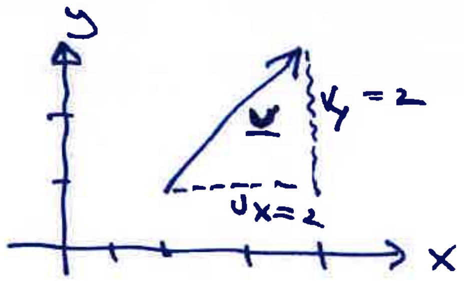
Plan

1. Vectors and vector operations
2. Span and linear independence
3. Vector spaces and dimension

Review:
- How to solve linear systems by Gaussian elimination
- Rank of a matrix
- Connection between pivot positions, rank and solutions of linear sys.

① Vectors and vector operations

Vector: has both magnitude and direction



Express $\underline{v}$ in coordinates:

$$\underline{v} = (v_x, v_y) = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

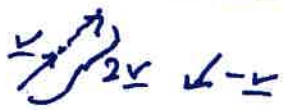
Vector operations:

Addition: $\underline{v} + \underline{w} = (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$

Scalar multiplication: $r \cdot \underline{v} = r \cdot (v_1, v_2, \ldots, v_n) = (rv_1, rv_2, \ldots, rv_n)$

Addition:

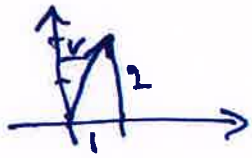
Scalar multiplication:



Length of a vector: $\|\underline{v}\| = \sqrt{v_1^2+v_2^2+\dots+v_n^2}$

(represents magnitude)

$\underline{v} = (1,2)$



$\|\underline{v}\| = \sqrt{1^2+2^2} = \sqrt{5}$

Inner product: (dot product)

$\underline{u}\cdot\underline{v} = \langle \underline{u},\underline{v}\rangle = u_1\cdot v_1 + u_2\cdot v_2 + \dots + u_n\cdot v_n$

(result is a number)

$\underline{u} \perp \underline{v}$ ($\underline{u}$ and $\underline{v}$ are orthogonal, i.e. there is a straight angle between $\underline{u}$ and $\underline{v}$) $\iff$ $\underline{u}\cdot\underline{v} = 0$

(2) Span and linear independence

$\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r$ vectors in $\mathbb{R}^n$; n-vectors (each have n components)

Defn: A linear combination of $\underline{v}_1,\dots,\underline{v}_r$ is an expression of the form

$$c_1\cdot\underline{v}_1 + c_2\cdot\underline{v}_2 + \dots + c_r\cdot\underline{v}_r$$

for some scalars (numbers) $c_1, c_2, \dots, c_r$.

Ex:

$\underline{v}_1 = (1,2,1)$

$v_2 = (3,-1,0)$

$c_1\cdot(1,2,1) + c_2\cdot(3,-1,0)$

$= (c_1, 2c_1, c_1) + (3c_2, -c_2, 0)$

$= (c_1+3c_2,\ 2c_1-c_2,\ c_1)$

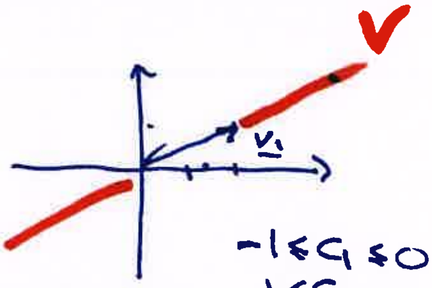
Defn: The span of $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r$ is the set of all linear combinations. We write:

$$V = \text{span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r)$$

Ex:

1) $\underline{v_1} = (2,1)$: $V = \text{span}(\underline{v_1})$

$\underline{v} = c_1 \cdot \underline{v_1}$

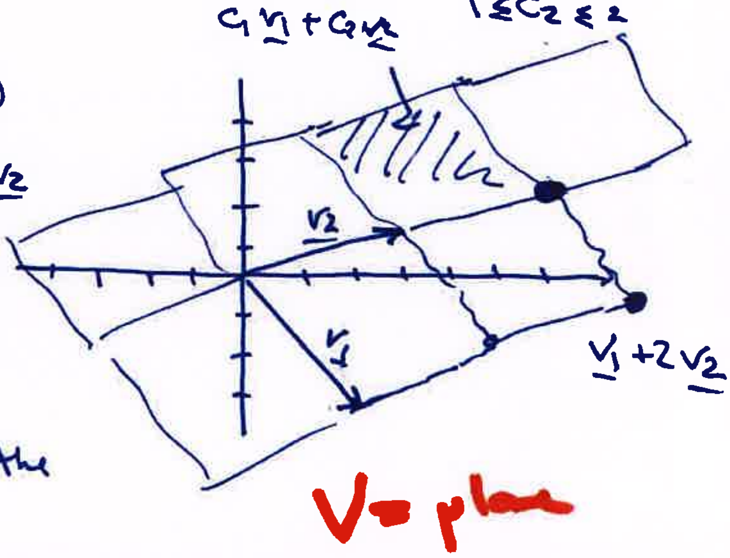


2) $\underline{v_1} = (3,1)$ $V = \text{span}(\underline{v_1}, \underline{v_2})$

$\underline{v_2} = (2,-3)$ $\underline{v} = c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2}$

$V = \mathbb{R}^2$

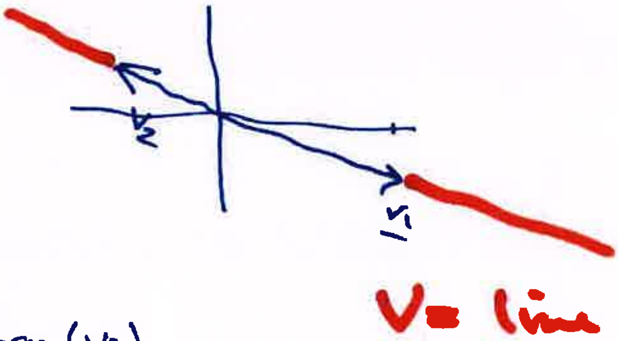
all pts in the plane



3) $\underline{v_1} = (6,-2)$

$\underline{v_2} = (-3,1)$

$-2\underline{v_2} = \underline{v_1}$



$V = \text{span}(\underline{v_1}, \underline{v_2}) = \text{span}(\underline{v_2})$

$c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} = c_1 \cdot (-2\underline{v_2}) + c_2 \cdot \underline{v_2}$

$= (-2c_1 + c_2)\underline{v_2}$

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Linear independence:

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ (vectors in $\mathbb{R}^n$).

Defn: The vectors $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ are linearly dependent if at least one of the vectors is a linear combination of the others.

Ex 3: $\underline{v_1}, \underline{v_2}$

$\underline{v_1} = -2\underline{v_2}$

$\underline{v_2} = -\frac{1}{2}\underline{v_1}$ $\quad \underline{v_1} + 2\underline{v_2} = \underline{0}$ $\longleftarrow$ linear dependency relation

Otherwise, the vectors are linearly independent.

Ex 2: $\underline{u_1} = (3,1)$

$\underline{u_2} = (2,-3)$

Proposition:

The vectors $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ are linearly independent if and only if the vector equation

$$\boxed{x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2} + \cdots + x_r \cdot \underline{v_r} = \underline{0}}$$

has only the trivial solution $(x_1, x_2, \ldots, x_r) = (0, 0, \ldots, 0)$.

Ex: $\underline{v_1} = (2,1,4)$

$\underline{v_2} = (3,5,-1)$

$\underline{v_3} = (5,-1,17)$

$$x_1 \cdot (2,1,4) + x_2 \cdot (3,5,-1) + x_3 (5,-1,17) = (0,0,0)$$

$$x_1 \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 5 \\ -1 \\ 17 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2x_1 + 3x_2 + 5x_3 \\ x_1 + 5x_2 - x_3 \\ 4x_1 - x_2 + 17x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 0 \\ x_1 + 5x_2 - x_3 &= 0 \\ 4x_1 - x_2 + 17x_3 &= 0 \end{aligned}}$$

homogeneous linear system

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 5 & -1 \\ 4 & -1 & 17 \end{pmatrix} = \left(\underline{v_1} \middle| \underline{v_2} \middle| \underline{v_3} \right)$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 1 & 5 & -1 \\ 4 & -1 & 17 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & -1 \\ 2 & 3 & 5 \\ 4 & -1 & 17 \end{pmatrix} \begin{matrix} \\ -2 \\ -4 \end{matrix} \to \begin{pmatrix} 1 & 5 & -1 \\ 0 & -7 & 7 \\ 0 & -21 & 21 \end{pmatrix} \begin{matrix} \\ \\ -3 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 5 & -1 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

x_3 free $\Rightarrow$ inf. many solutions

$\Rightarrow$ $\underline{v}_1, \underline{v}_2, \underline{v}_3$ lin. dependent

$$x_1 + 5x_2 - x_3 = 0$$
$$-7x_2 + 7x_3 = 0$$

$$x_1 = -5x_2 + x_3 = -4x_3$$
$$x_2 = x_3$$
$$\Downarrow$$
$$(x_1, x_2, x_3) = (-4x_3, x_3, x_3) \quad x_3 \text{ free}$$
$$= x_3 \cdot (-4, 1, 1)$$

$\underline{x_3 = 1}$: $\quad x_1 = -4 \quad x_2 = 1 \quad x_3 = 1$

$$\Downarrow$$

$$-4\underline{v}_1 + 1 \cdot \underline{v}_2 + 1 \cdot \underline{v}_3 = \underline{0}$$

linear dependency relation

$$\underline{v}_3 = 4\underline{v}_1 - \underline{v}_2$$

$$V = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \text{span}(\underline{v}_1, \underline{v}_2)$$

3 Vector spaces and dimension

$\mathbb{R}^n$ = the set of all vectors in n-dim. space = Euclidean n-space

$\underline{v} = (v_1, v_2, \ldots, v_n)$

This is the model for all vector spaces.

is a vector space:

- a collection of vectors
- with operations (addition, scalar multiplication)

satisfying certain laws for vectors

What other vector spaces are there?

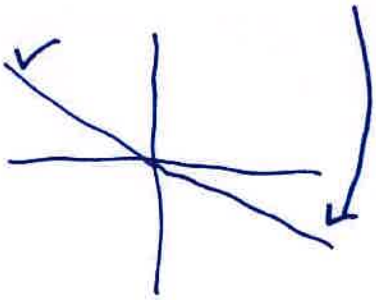
If V is a subset of $\mathbb{R}^n$ (for example a line, a plane, etc) it is a vector space if

i) 0 is in V

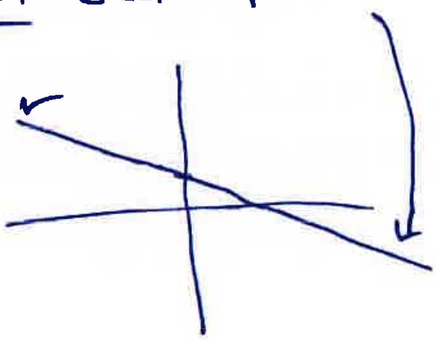
ii) $\underline{v}, \underline{w}$ are in $V \Rightarrow \underline{v} + \underline{w}$ in V

iii) $\underline{v}$ in $V \Rightarrow r \cdot \underline{v}$ in V for all scalars r

Ex: Solutions of $x + 2y = 0$ in $\mathbb{R}^2$ is a vector space



Solutions of $x + 2y = 1$ not vector space



Important examples:

a) $V = \text{span}(\underline{v}_1, \dots, \underline{v}_r)$ is always a vector space

b) $V = \text{Col}(A)$ — " —

c) $V = \text{Null}(A)$ — " —

Column space:

$A = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_r)$: $\boxed{\text{col}(A) = \text{span}(\underline{v}_1, \dots, \underline{v}_r)}$

m×r matrix

$\underline{v}_1, \dots, \underline{v}_r$ are the columns of A

the column space of a matrix is the span of its column vectors

col(A) ⟺ span($\underline{v}_1, \dots, \underline{v}_r$)

two ways of writing the same thing.

(a) and (b) above

Ex: $A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 5 & -1 \\ 4 & -1 & 12 \end{pmatrix}$ $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 5 \\ -1 \\ 12 \end{pmatrix}$

$V = \text{Col}(A) = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ ← see computation earlier in the lecture

$= \text{span}(\underline{v}_1, \underline{v}_2)$

$\boxed{\underline{v}_3 = 4\underline{v}_1 - \underline{v}_2}$

Want to find a minimal set of vectors that span V. It is called basis of V (see defn. next page).

Defn:

V general vector space

A base of V is a set $\{\underline{v}_1, \dots, \underline{v}_k\} = \mathcal{B}$ of vectors such that

i) $\text{Span}(\mathcal{B}) = \text{span}(\underline{v}_1, \dots, \underline{v}_k) = V$

ii) $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_k\}$ are linearly independent

The dimension is

$$\boxed{\dim(V) = \#\text{ vectors in a base}}$$

Proposition:

Let A be any matrix and $V = \text{Col}(A)$. Then:

i) the columns in A corresponding to pivots form a base for V

ii) $\dim V = \text{rk}(A)$

Ex:

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 5 & -1 \\ 4 & 7 & 7 \end{pmatrix} = (\underline{v}_1 | \underline{v}_2 | \underline{v}_3)$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 5 & -1 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

echelon form

pivots

$\rightsquigarrow$ $V = \text{Col}(A) = \text{Span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ has base

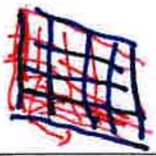
$\mathcal{B} = \{\underline{v}_1, \underline{v}_2\}$

and

$\dim V = \text{rk}\, A = \underline{\underline{2}}$

\# pivots

The column space or span is 2-dim, i.e. a plane in 3-dim space $\mathbb{R}^3$ that goes through $(0,0,0)$



Col(A)

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Nullspace:

A $m \times n$ matrix

$\text{Null}(A) = \left\{ \begin{array}{l} \text{all solutions of the linear} \\ \text{system that is homogeneous} \\ \text{and has coeff. matrix } A \end{array} \right\}$

Ex: $A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 5 & -1 \\ 4 & -1 & 17 \end{pmatrix} \rightsquigarrow V = \text{Null}(A) = \text{all solutions of}$

$$\begin{aligned} 2x + 3y + 5z &= 0 \\ x + 5y - z &= 0 \\ 4x - y + 17z &= 0 \end{aligned}$$

Solve using Gauss (see earlier in the lecture):

$(x,y,z) = (-4z, z, z)$ with z free

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4z \\ z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} = z \cdot \underline{w_1}$$

all multiples of $\underline{w_1} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$

$$\boxed{\text{Null}(A) = \text{Span}(\underline{w_1})}$$

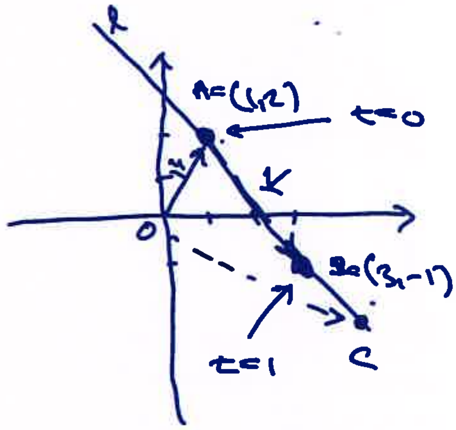
Result: A $m \times n$ matrix

$\text{Null}(A)$ has a base that can be found using Gaussian elimination and it has

$$\boxed{\dim \text{Null}(A) = \underbrace{n - rk(A)}_{\#\text{ free variables}}}$$

Plan: (1) Parametrization of lines and planes
(2) Distance of points
(3) Inner products, lengths and projection
(4) An example: Null(A)

(1) Parametrization of a line



Ex: Line l thr. $(1,2)$, $(3,-1)$

$OA = (1,2) = \underline{u}$

$AB = (2,-3) = \underline{v}$

$$OC = OA + AC = OA + AB\cdot t$$
$$= \underline{u} + t\cdot\underline{v} = (1,2) + t\cdot(2,-3)$$
$$= (1+2t, 2-3t)$$

$$C: \quad (x,y) = (1+2t, 2-3t)$$

(2) Distance between points:

Ex: $A = (1,2)$, $B = (3,-1)$

What is $d(A,B)$? (distance between A and B)

$\underline{v} = (2,-3)$

$$\|\underline{v}\| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

$$d(A,B) = \sqrt{13}$$

Formula: $P = (a_1, a_2, \dots, a_n)$, $Q = (b_1, b_2, \dots, b_n)$

$$d(P,Q) = \|(b_1-a_1, b_2-a_2, \dots)\| = \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + \dots + (b_n-a_n)^2}$$

(3) Inner products, length and projections

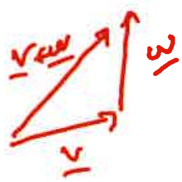
$$\underline{u}\cdot\underline{w} = (u_1,\dots,u_n)\cdot(v_1,\dots,v_n) = u_1\cdot v_1 + u_2\cdot v_2 + \dots + u_n\cdot v_n$$

$$\underline{v}\cdot\underline{v} = v_1^2 + v_2^2 + \dots + v_n^2 \qquad \|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

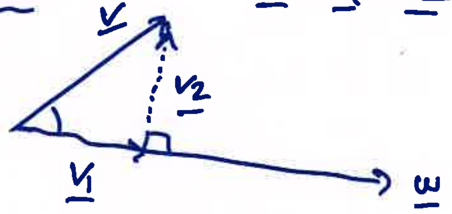
$$\boxed{\underline{v}\cdot\underline{v} = \|\underline{v}\|^2}$$

$$(\underline{v}+\underline{w})\cdot(\underline{v}+\underline{w}) = \underline{v}\cdot\underline{v} + \underline{w}\cdot\underline{v} + \underline{v}\cdot\underline{w} + \underline{w}\cdot\underline{w}$$

$$\|\underline{v}+\underline{w}\|^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2 + 2(\underline{v}\cdot\underline{w})$$



Projections:



$\underline{v} = \underline{v_1} + \underline{v_2}$

$$\boxed{\text{proj}_{\underline{w}}(\underline{v}) = \underline{v_1} = \frac{\underline{v}\cdot\underline{w}}{\underline{w}\cdot\underline{w}}\cdot\underline{w}}$$

$\underline{v_1} = r\cdot\underline{w}$

$\underline{v_1} \perp \underline{v_2} \iff \underline{v_1}\cdot(\underline{v_2}) = 0$

$$r\underline{w}\cdot(\underline{v} - \underline{v_1}) = 0$$
$$r\underline{w}\cdot(\underline{v} - r\underline{w}) = 0$$
$$r\cdot(\underline{v}\cdot\underline{w}) - r^2\cdot\underline{w}\cdot\underline{w} = 0$$
$$\underline{v}\cdot\underline{w} - r(\underline{w}\cdot\underline{w}) = 0$$
$$\boxed{r = \frac{\underline{v}\cdot\underline{w}}{\underline{w}\cdot\underline{w}}}$$

(4) **An example:** Null(A)

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \end{pmatrix} \begin{matrix} \\ ]{-1} \\ ]{-2} \end{matrix}$$

Null(A): Solution of:

$$\boxed{\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 - x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 \quad\quad + 2x_4 &= 0 \end{aligned}}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix}]{-1} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_1, x_2: basic
x_3, x_4: free

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_2 - 2x_3 \quad\quad &= 0 \end{aligned}$$

$$\begin{aligned} x_2 &= 2x_3 \\ x_1 &= -x_2 - x_3 - x_4 \\ &= -2x_3 - x_3 - x_4 \\ &= -3x_3 - x_4 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_3 - x_4 \\ 2x_3 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_3 \\ 2x_3 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{pmatrix} = x_3\cdot\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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$$\underline{w_1} = \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} \quad \underline{w_2} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} : \quad \text{Null}(A) = \text{Span}(\underline{w_1}, \underline{w_2})$$

$$\dim \text{Null}(A) = 2$$

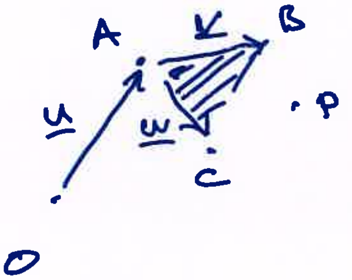
\# free vars.

vectors resulting from solving homog. lin. sys. using Gauss are always linearly independent

$$\boxed{\dim \text{Null}(A) = n - rk(A)}$$

Base: $\{\underline{w_1}, \underline{w_2}\}$ for Null(A)

1) Parametrization of a plane



A, B, C are three pts

Defn: A, B, C collinear if they lie on a straight line

Assume that this is not the case

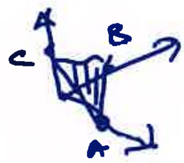
P in the plane thr. A, B, C:

$$OP = OA + s \cdot AB + t \cdot AC$$

$A = (a_1, \ldots, a_n)$
$B = (b_1, \ldots, b_n)$
$C = (c_1, \ldots, c_n)$

$$\boxed{(x_1, \ldots, x_n) = (a_1, \ldots, a_n) + s \cdot (b_1 - a_1, \ldots, b_n - a_n) + t\,(c_1 - a_1, \ldots, c_n - a_n)}$$

Ex: $A = (1,0,0)$
$B = (0,1,0)$
$C = (0,0,1)$



$$(x, y, z) = (1,0,0) + s \cdot (-1,1,0) + t\,(-1,0,1)$$
$$= \underline{\underline{(1-s-t,\ s,\ t)}}$$