Plan

1. Introduction to first order differential equations
2. Separable differential equations
3. Linear first order differential equations
4. Exact differential equations

Plenary Session 3: Monday at 17-20 in A1-040
Problems: Lecture 7-9

Review:

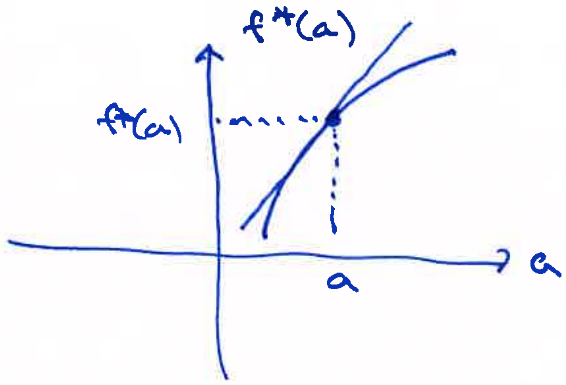
- envelope thm (unconstrained / constrained case)

max/min $f(\underline{x})$ (parameter a)

$$\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))$$

max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) - a_1 = 0 \\ \vdots \\ g_m(\underline{x}) - a_m = 0 \end{cases}$ $(\leq)$ (parameter a)

$$\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \underline{\lambda}^*(a))$$



① Introduction to first order differential equations

Defn: A first order diff. eqn. is an equation where the unknown is a function $y = y(t)$, which involves the first order derivative $y' = y'(t)$.

Ex: $\boxed{y' = 2t} \Leftrightarrow y'(t) = 2t$

$y = \int 2t\,dt = t^2 + C$

$y(t) = \underline{\underline{t^2 + C}}$ — general solution of the differential eqn.

C: undetermined coeff.

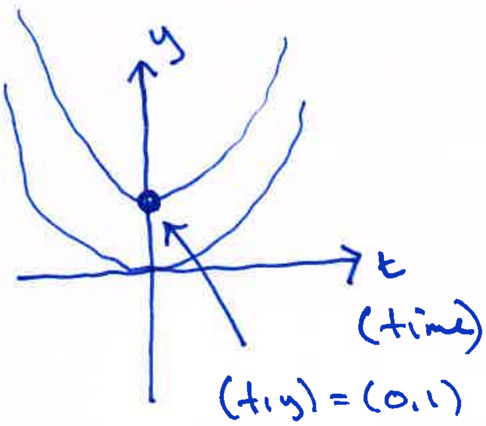
$y' = 2t, \quad y(0) = 1$ ← initial value problem

$y = t^2 + C \qquad y(0) = 1: \quad t = 0 \Rightarrow y = 1$

$1 = 0^2 + C$

$\underline{C = 1}$

$y = \underline{\underline{t^2 + 1}}$ — particular solution



Ex: $ty' = 2y \Rightarrow y' = \dfrac{2y}{t}$

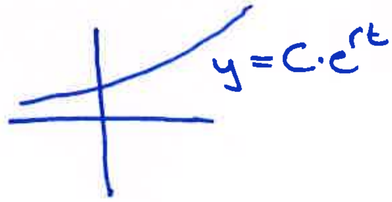
General form of "most" first order differential equations: $\boxed{y' = F(t,y)}$

Idea: We use differential equations to model change

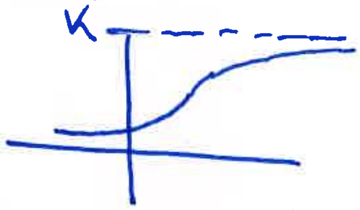
For example:

i) $y' = r \cdot y$ $\quad$ (r constant)

growth rate of y



ii) $y' = ry(1 - y/K)$ $\quad$ (r, K constants)



logistic "S-curve"

② Separable differential equations

Defn: A first order diff.-eqn. is called separable if it can be written

$$y' = f(t) \cdot g(y)$$

In general: $y' = F(t,y)$

Ex: $y' = 2y = \underbrace{2}_{f(t)} \cdot \underbrace{y}_{g(y)}$ $\quad$ separable

$$y' + 2y = yt$$
$$y' = yt - 2y$$
$$y' = \underbrace{(t-2)}_{f(t)} \cdot \underbrace{y}_{g(y)}$$ separable

$$y' + y = t$$
$$y' = t - y$$ not separable

Solution method (separable diff. eqn.)

Ex: $y \cdot y' = t$

$$y' = \frac{t}{y} = \underbrace{t}_{f(t)} \cdot \underbrace{\frac{1}{y}}_{g(y)}$$ separable $| \cdot y$

$$yy' = t$$ ← separated form $| \int - \, dt$

$$\int yy' \, dt = \int t \, dt$$

$y' dt = dy$: substitution $\quad u = y(t), \; du = y' \cdot dt$

$$\int y \, dy = \tfrac{1}{2}t^2 + C$$

$$\tfrac{1}{2}y^2 + C_1 = \tfrac{1}{2}t^2 + C_2$$ ← solution in implicit form

$$\tfrac{1}{2}y^2 = \tfrac{1}{2}t^2 + C_2 - C_1 \quad | \cdot 2$$

$$y^2 = t^2 + 2(C_2 - C_1)$$

$K = 2(C_2 - C_1)$

$$y = \pm\sqrt{t^2 + 2(C_2 - C_1)}$$

$$\underline{\underline{y = \pm\sqrt{t^2 + K}}}$$ general solution in explicit form

Alternative: $\int y y' \, dt = \int t \, dt$

$$\int y \frac{dy}{\cancel{dt}} \cdot \cancel{dt} = \int t \, dt$$

$$\int y \, dy = \int t \, dt$$

In general:

$$y' = f(t)\cdot g(y) \qquad | : g(y)$$

$$\frac{1}{g(y)} y' = f(t) \qquad | \int - \, dt$$

$$\int \frac{1}{g(y)} y' \, dt = \int f(t)\, dt \qquad y' dt = dy$$

$$\boxed{\int \frac{1}{g(y)} \, dy = \int f(t) \, dt}$$

$\downarrow$ solve both integrals

Implicit solution

$\downarrow$ solve for y

Explicit general solution

Ex:

$$y' = ry \qquad (r \text{ constant}) \qquad | : y$$

$$\frac{1}{y} y' = r \qquad | \int - \, dt$$

$$\int \frac{1}{y} y' \, dt = \int r \, dt$$

$$\int \frac{1}{y} \, dy = \int r \, dt$$

$$\ln|y| + C_1 = rt + C_2$$

$$\ln|y| = rt + C_2 - C_1 \qquad | e^{\cdot}$$

$$e^{\ln|y|} = e^{rt + C_2 - C_1}$$

$$|y| = e^{rt} \cdot e^{C_2 - C_1}$$

$$y = (\pm e^{C_2 - C_1}) \cdot e^{rt} \qquad y = Ke^{rt} \quad \text{general solution (explicit)}$$

2 Linear first order differential equations

Defn.: A first order differential equation is linear if it can be written

$$\boxed{y' + a(t)y = b(t)} \iff y' = \underbrace{b(t) - a(t)y}_{\text{linear in } y}$$

Ex.:

$y' + 2y = t^2$ $\quad \begin{cases} a(t) = 2 \\ b(t) = t^2 \end{cases}$ linear

$y' - y = t$ $\quad \begin{cases} a(t) = -1 \\ b(t) = t \end{cases}$ linear

$y' + 3y = 2$ $\quad \begin{cases} a(t) = 3 \\ b(t) = 2 \end{cases}$ linear, const. coeff.

Solution method: Integrating factor (works for all linear first order diff. eqn.)

Ex: $y' - y = t \quad | \cdot u(t)$ ← $u(t)$: integrating factor

$u \cdot y' - uy = u \cdot t \quad | \int - \, dt$

$\int (uy' - uy)\,dt = \int u(t)t\,dt$

$\int (uy)'\,dt = \int t\,u(t)\,dt$

$uy = \int t\,u(t)\,dt$

$y = \frac{1}{u(t)} \cdot \int t \cdot u(t)\,dt$

We want:

$uy' - uy = (uy)'$

$uy' - uy = uy' + u' \cdot y$

$-u = u'$

$u' = -1 \cdot u$

$\boxed{u = Ke^{-t}}$

Conclusion: We choose $\underline{u = e^{-t}}$ as integrating factor.

$$y' - y = t \qquad | \cdot e^{-t}$$

$$e^{-t} y' - e^{-t} \cdot y = t e^{-t}$$

$$(e^{-t} \cdot y)' = -e^{-t} \cdot y + e^{-t} \cdot y'$$

$$(e^{-t} \cdot y)' = t e^{-t}$$

$$e^{-t} \cdot y = \int t e^{-t} \, dt$$

$$y = e^{t} \cdot \int t e^{-t} \, dt$$

$$y = e^{t} \left(-t e^{-t} - e^{-t} + C \right)$$

$$\underline{y = -t - 1 + C e^{t}}$$

general solution

$$\int t \cdot e^{-t} \, dt = \quad \leftarrow \text{int. by parts}$$

$$\int u' v \, dt = uv - \int u v' \, dt$$

$$u = -e^{-t} \quad v = t$$
$$u' = e^{-t} \quad v' = 1$$

$$= -t e^{-t} - \int -e^{-t} \cdot 1 \, dt$$

$$= -t e^{-t} + \int e^{-t} \, dt$$

$$= \underline{-t e^{-t} - e^{-t} + C}$$

In general:

$$y' + a(t) \cdot y = b(t)$$

$$u y' + u \cdot a(t) y = b(t) u(t)$$

$$(u y)' = b(t) u(t) \qquad | \int - \, dt$$

$$u y = \int b(t) u(t) \, dt$$

$$\boxed{y = \frac{1}{u(t)} \cdot \int b(t) u(t) \, dt}$$

Integrating factor:

$$\boxed{u(t) = e^{\int a(t) \, dt}}$$

Ex: $ty' = 2y \implies y' = \frac{2y}{t} = \frac{2}{t} \cdot y$ separable

$y' - \frac{2}{t}y = 0$ linear

$a(t) = -\frac{2}{t}$

$b(t) = 0$

Integrating factor:

$a(t) = -\frac{2}{t} \implies \int a(t)\,dt = \int -\frac{2}{t}\,dt$

$= -2\int \frac{1}{t}\,dt = -2\ln|t| + C$

$u = e^{\int a(t)\,dt} = e^{-2\ln|t|}$

$= e^{\ln|t| \cdot (-2)} = \left(e^{\ln|t|}\right)^{-2}$

$= |t|^{-2} = \frac{1}{|t|^2} = \frac{1}{t^2}$

$u = \frac{1}{t^2}$

$y' - \frac{2}{t}y = 0 \quad \cdot \frac{1}{t^2}$

$\frac{1}{t^2}y' - \frac{2}{t^3}y = 0$

$\left(\frac{1}{t^2} \cdot y\right)' = 0$

$\frac{1}{t^2}y = C$

$y = \underline{Ct^2}$ general solution

3 Exact differential equations

Defn: A first order differential equation is exact if it can be written

$$p(t,y) + q(t,y)\cdot y' = 0$$

where $\left\{ \boxed{\begin{array}{l} p(t,y) = h'_t \\ q(t,y) = h'_y \end{array}} \right\}$ for some function $h(t,y)$ in two variables.

Ex: $2t + 2y\cdot y' = 0$ exact form

$\left\{ \begin{array}{l} h'_t = 2t \\ h'_y = 2y \end{array} \right\}$ $h = t^2 + y^2$

ok, it is exact

Solution method:

$$p(t,y) + q(t,y)\cdot y' = 0$$
$$h'_t + h'_y\cdot y' = 0$$
$$\Updownarrow$$
$$\boxed{h(t,y) = C}$$

Ex: $2t + 2y\cdot y' = 0$

$h'_t + h'_y\cdot y' = 0$

total derivative: $\frac{dh}{dt} = \frac{d}{dt}(t^2+y^2)$

$= 2t + 2y\cdot y'$

$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y}\cdot\frac{dy}{dt}$

$= h'_t + h'_y\cdot y' = 0$

$\frac{dh}{dt} = 0 \Leftrightarrow h(t,y) = C$

$h = t^2 + y^2$

$h = t^2 + y^2 = C$ implicit solution

$y^2 = C - t^2$

$y = \pm\sqrt{C - t^2}$ general solution expl. form

Ex: $(3t^2-3y)+(3y^2-3t)y'=0$ $\qquad p+q\cdot y'=0$

Exact form? Yes

$h'_t = 3t^2-3y = p$
$h'_y = 3y^2-3t = q$

looking for h that is a solution of these two equations

Test: $\boxed{\begin{array}{l} p'_y = q'_t \iff \text{there is an } h \text{ s.t. } h'_t = p,\ h'_y = q \\ \quad \shortparallel \qquad \shortparallel \\ h''_{ty} = h''_{yt} \end{array}}$

$p'_y = (3t^2-3y)'_y = -3$
$q'_t = (3y^2-3t)'_t = -3$

there exists an h (it is an exact form)

Find h: $\boxed{\begin{array}{l} h'_t = 3t^2-3y \\ h'_y = 3y^2-3t \end{array}}$ $\Rightarrow$ (1) $h'_t = 3t^2-3y+0$

$h = \int 3t^2-3y \, dt$

$h = \underline{t^3-3yt+Q(y)}$ $= y^3$

(2) $h'_y = 3y^2-3t$

$(t^3-3yt+Q(y))'_y = 3y^2-3t$

$\cancel{0} - 3t + Q'(y) = 3y^2 - 3t$

$Q'(y) = 3y^2$

$Q(y) = y^3$

$h = t^3-3yt+y^3 = C$

$\underline{\underline{y^3-3ty+t^3=C}}$ $\qquad$ $\boxed{\text{Solution in implicit form}}$

Ex: $y' = 5y(1 - y/10)$

(separable) $= \underbrace{5}_{f(t)} \cdot \underbrace{y(1 - y/10)}_{g(y)}$

$y' = ry \cdot (1 - y/K)$

$$\frac{1}{y(1 - y/10)} y' = 5$$

$$\int \frac{1}{y(1-y/10)} y' \, dt = \int 5 \, dt$$

$$\int \frac{10}{y(10-y)} \, dy = 5t + C$$

Partial fractions:

$$\frac{10}{y(10-y)} = \frac{A^{=1}}{y} + \frac{B^{=1}}{10-y} \quad | y(10-y)$$

$$10 = A \cdot (10-y) + By$$

$$10 = (-A+B)y + (10A)$$

$$\boxed{\begin{matrix} -A+B=0 \\ 10A = 10 \end{matrix}} \quad \begin{matrix} B=1 \\ A=1 \end{matrix}$$

$$\int \frac{1}{y} + \frac{1}{10-y} \, dy = 5t + C$$

$$\ln|y| - \ln|10-y| = 5t + C$$

$$\ln \frac{|y|}{|10-y|} = 5t + C \quad | e^{\cdot}$$

$$\boxed{\begin{matrix} \ln a + \ln b = \ln(a \cdot b) \\ \ln a - \ln b = \ln(a/b) \end{matrix}}$$

$$\left| \frac{y}{10-y} \right| = \frac{|y|}{|10-y|} = e^{5t+C} = e^{5t} \cdot e^{C}$$

$$\frac{y}{10-y} = \pm e^{C} e^{5t} = K \cdot e^{5t} \quad | \cdot (10-y)$$

$$y = (10-y) \cdot Ke^{5t} = 10Ke^{5t} - y \cdot Ke^{5t}$$

$$y + y \cdot Ke^{5t} = 10Ke^{5t}$$

$$\frac{y \cdot (1 + Ke^{5t})}{1 + Ke^{5t}} = \frac{10Ke^{5t}}{1 + Ke^{5t}}$$

$$y = \frac{10Ke^{5t}}{1 + Ke^{5t}} \qquad \text{general solution}$$

$$= 10 \cdot \frac{Ke^{5t}}{1 + Ke^{5t}}$$

$\searrow 1$ as $t \to \infty$