Plan

1. Linear systems and geometry
2. Gaussian elimination
3. Rank of a matrix

Lecturer: Eivind Eriksen
Office B4-032
Office hours: Thu 12-14
+ open door at other times
Prefered email:
eivind.eriksen@bi.no

FORK 1003: Prep. course mathematics
Lecture 1.

1 Linear systems and geometry

Ex:

$$\boxed{\begin{aligned} x+y+z+w &= 7 \\ x-y+2z+3w &= 13 \\ 2x+3y \quad -w &= 5 \end{aligned}}$$

3x4 linear system

$V = \{(x,y,z,w) : \text{solutions of lin. sys.}\}$

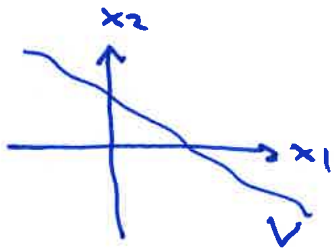
all pts. (x,y,z,w) that are solutions

Defn: Linear equation $\boxed{a_1x_1 + a_2x_2 + \dots + a_nx_n = b}$ for given numbers $a_1, a_2, \dots, a_n, b$

It is called degenerate if $a_1 = a_2 = \dots = a_n = 0$.

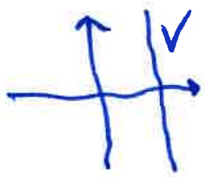
$n=2$: $a_1x_1 + a_2x_2 = b$ $\xRightarrow{a_2 \neq 0}$ $\dfrac{a_2x_2}{a_2} = \dfrac{b - a_1x_1}{a_2}$

$x_2 = \dfrac{b}{a_2} - \dfrac{a_1}{a_2}x_1$



$a_2 = 0$: $\dfrac{a_1x_1}{a_1} = \dfrac{b}{a_1}$ ($a_1 \neq 0$)

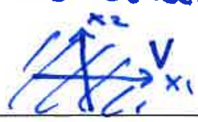
$x_1 = b/a_1$



$a_2 = 0,\ a_1 = 0$: $0 \cdot x_1 + 0 \cdot x_2 = b$

$b \neq 0$: no solution

$b = 0$:



$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ $\rightsquigarrow$ V is a line, plane,
linear eqn. (n=2) (n=3)
non-degenerate

a hyperplane
(in general)

Defn: A hyperplane is the set V of all solutions of one linear equation


n=2 n=3

Defn: An $m \times n$ linear system (of equations) is a system of m equations (that are linear) in n variables:

$$m \left\{ \boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}} \right.$$

n variables

$a_{11}, \dots, a_{mn}, b_1, \dots, b_m$ are given numbers

Result:

① If $P \neq Q$ are two different points on a hyperplane H, then the line through P and Q lies in H.

② For any $m \times n$ linear system, there are either

i) no solution } inconsistent
ii) one solution } consistent
iii) infinitely many solutions } consistent

(2) Gaussian elimination

We start with an $m \times n$ linear system.

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \qquad (A|\underline{b}) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

coeff. matrix   augmented matrix

Gaussian elimination: general method for solving linear systems

i) Write down the augmented matrix
ii) Use elementary row operations until you get an echelon form } Gaussian process
iii) Write the echelon form back to a system of equations, and solve it by back substitution.

Elementary row operations:
i) Switch two rows
ii) Multiply a row with $c \neq 0$
iii) Add a multiple of one row to another row.

Echelon form:
Pivot: the first non-zero number in a row
Echelon form: A matrix such that
i) All zero rows are below any non-zero row
ii) All entries under a pivot are zero.

Ex:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 3 & 13 \\ 2 & 3 & 0 & -1 & 5 \end{array}\right) \xrightarrow{\;R_2 + (-1)R_1,\ R_3 + (-2)R_1\;} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 & 6 \\ 0 & 1 & -2 & -3 & -9 \end{array}\right)$$

get zero

aug. matrix

get zero

$$\xrightarrow{\;R_3 + \frac{1}{2}R_2\;} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 & 6 \\ 0 & 0 & -3/2 & -2 & -6 \end{array}\right)\cdot 2 \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 & 6 \\ 0 & 0 & -3 & -4 & -12 \end{array}\right)$$

echelon form echelon form

Defn: Two matrices are <u>row equivalent</u> if you can get from one to the other using elementary row operations.

Result: 1) Any matrix is row equivalent with an echelon form. The echelon form is <u>not unique</u>.

2) Any matrix is row equivalent to a <u>unique</u> <u>reduced echelon form</u>.

Defn: A <u>reduced echelon form</u>: An echelon form that in addition satisfies:

iii) All pivots are 1

iv) All entries <u>over</u> a pivot are zero.

$\boxed{\text{Gaussian process} \Downarrow \text{reduced echelon form}}$ Gauss-Jordan elimination (variation of Gaussian el.) (optional to use this)

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 & 6 \\ 0 & 0 & -3 & -4 & -12 \end{array}\right)$$

echelon form

$$\begin{aligned} \underline{x}+y+z+w &= 7 \\ \underline{-2y}+z+2w &= 6 \\ \underline{-3z}-4w &= -12 \end{aligned}$$

Back substitution:

$-3\underline{z}-4w=-12$

$$\frac{-3z}{-3}=\frac{-12+4w}{-3}$$

$$z = \underline{4-\tfrac{4}{3}w}$$

$-2\underline{y}+z+2w=6$

$$-2y = 6-\left(4-\tfrac{4}{3}w\right)-2w$$

$$\frac{-2y}{-2} = \frac{2-\frac{2}{3}w}{-2}$$

$$y = \underline{-1+\tfrac{1}{3}w}$$

$\underline{x}+y+z+w=7$

$$x = 7-\left(-1+\tfrac{1}{3}w\right)-\left(4-\tfrac{4}{3}w\right)-w$$

$$= \underline{4}$$

Conclusion: $(x,y,z,w)=\underline{\left(4,-1+\tfrac{1}{3}w,\ 4-\tfrac{4}{3}w,\ w\right)}$

where w is free

basic variables: x, y, z (pivots in variable column)

free variables: w (no pivot in var. col.)

In this case: one degree of freedom (one free var.), infinitely many solutions

③ Rank of a matrix

Def: A ($m \times n$ matrix) $\rightsquigarrow$ $rk(A)$ (rank of A) $=$ # pivot positions in an echelon form of A

$A = 0$ (zero matrix) $\Rightarrow rk(A) = 0$

$A \neq 0$ (at least one entry is non-zero) $\Rightarrow 1 \leq rk(A) \leq \min\{m, n\}$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 3 & 13 \\ 2 & 3 & 0 & -1 & 5 \end{pmatrix}$ $\quad rk(A) = \underline{3}$

Theorem: We consider an $m \times n$ linear system with coefficient matrix A and augmented matrix $(A|b)$. Then we have:

i) The linear system is consistent if and only if $rk\, A = rk\,(A|b)$.

ii) In case it is consistent, then:

$n - rk(A) = $ # free variables
$=$ number of degrees of freedom
$= \dim(V)$, where V is the set of all solutions of the linear system

Explanation:

i) rk A = rk$(A|\underline{b})$:

no pivot in the $\underline{b}$-column

$$(A|\underline{b}) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

Ex: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right)$ rk$(A|\underline{b}) = 2$, rk$(A) = 1$

$0 \cdot x + 0 \cdot y + 0 \cdot z = 1$ means no solution

ii) $n - \text{rk}(A) = \#$ free variables

- $n - \text{rk}(A) = 0$: no free variables → one solution ($\text{rk}A = n$)
- $n - \text{rk}(A) > 0$: at least one free variable → infinitely many solutions

Interpretation: V = all solutions of the linear system

$$\boxed{\dim V = n - \text{rk}(A)}$$

$n - \text{rk}(A) = 0$

dim = 0

$n - \text{rk}(A) = 1$

dim = 1

$n - \text{rk}(A) = 2$:

dim = 2

Key Problems

Problem 1.
Use Gaussian elimination to solve the linear systems with the following augmented matrices:

a) $\left(\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 2 & -1 & 3 & 3 \\ 3 & 2 & 5 & 12 \end{array}\right)$ b) $\left(\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 2 & -1 & 3 & 3 \\ 3 & 2 & 7 & 12 \end{array}\right)$ c) $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 8 \\ 1 & 3 & 1 & 5 & 28 \\ 2 & 4 & 2 & 9 & 48 \end{array}\right)$

Problem 2.
Determine how many solutions the linear system has:

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 6 \\ x & + & 2y & + & 4z & = & 13 \\ x & + & 3y & + & 9z & = & 24 \end{array}$$

Does the number of solutions change if we change the blue coefficient in the first equation? In that case, determine how the number of solutions changes with the blue coefficient.

Problem 3.
We consider the homogeneous linear system with coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 4 & -1 \\ 5 & 5 & -1 & 4 \\ 7 & 6 & 3 & 3 \end{pmatrix}$$

Describe the set of solutions geometrically. How many degrees of freedom are there? Does this change if we change the red coefficient in the second row?

Problems from the Workbook and Lecture Notes

Exercise problems:	Eriksen [E] 1.1 - 1.16 (see It's Learning)
Optional problems:	Workbook [W] 1.1 - 1.18 (some problems are the same as the ones in [E])

Answers to Key Problems

Problem 1.
a) $(x,y,z) = (1,2,1)$ b) No solutions c) $(x,y,z,w) = (2-z, 2, z, 4)$ with z free

Problem 2.
There is one unique solution. The number of solutions only changes if the blue coefficient is -1, in which case there are no solutions. For any other value, there is a unique solution.

Problem 3.
We have that $\text{rk}(A) = 3$, and there is $n - \text{rk}(A) = 4 - 3 = 1$ degrees of freedom. Therefore the set of solutions is a straight line in $\mathbb{R}^4$. If we change the red coefficient, the rank of A remains $\text{rk}(A) = 3$ unless the coefficient is 6, in which case $\text{rk}(A) = 2$. Therefore, the set of solutions is a line for all values of the red coefficient except 6, and in this case the set of solutions is a plane since the dimension is $n - \text{rk}(A) = 4 - 2 = 2$.

Lecture 1 Part 2 (video lecture)

1. More examples
2. Homogeneous systems

Ex:

$$\boxed{\begin{aligned} x+y+z+w &= 5 \\ 2x-y+3z-w &= 2 \\ x+4y +3w &= 12 \end{aligned}}$$

3×4 lin. sys.

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 2 & -1 & 3 & -1 & 2 \\ 1 & 4 & 0 & 3 & 12 \end{array}\right) \begin{matrix} \\ ]-2 \\ ]-1 \end{matrix}$$

$$\begin{matrix} \\ -1 \end{matrix} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -3 & 1 & -3 & -8 \\ 0 & 3 & -1 & 2 & 7 \end{array}\right)], \rightarrow \begin{array}{c} \begin{matrix} x & y & z & w & \end{matrix} \\ \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -3 & 1 & -3 & -8 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right) \end{array}$$

echelon form

no pivot in b-col ⇕ consistent lin. sys.

$\boxed{\begin{aligned} &x, y, w : \text{basic} \\ &z : \text{free} \end{aligned}}$

$$\begin{aligned} x+y+z+w &= 5 \\ -3y+z-3w &= -8 \\ -w &= -1 \end{aligned}$$

$-w = -1 \Rightarrow \underline{w = 1}$

$-3y+z-3w = -8$

$-3y = -8 - z + 3 = -5 - z \quad \Big| : (-3)$

$\underline{y = 5/3 + 1/3 \cdot z}$

$x+y+z+w = 5$

$x = 5 - (5/3 + 1/3 \cdot z) - z - 1$

$\underline{x = 7/3 - 4/3 \cdot z}$

Solution: $\underline{(x, y, z, w) = \left(\frac{7}{3} - \frac{4}{3}z, \frac{5}{3} + \frac{1}{3}z, z, 1\right)}$ with z free

Comments: i) There are infinitely many solutions, one degree of freedom.

$$n - rk(A) = 4 - 3 = 1$$

In fact, $V = \{(x,y,z,w) : \text{solutions}\}$
$= \{(\frac{5}{3} - \frac{4}{3}z, \frac{7}{3} + \frac{1}{3}z, z, 1) : z \text{ free}\}$

V is a line in 4-dim space

dim V = 1
no curvature

ii) Free variables: When we use Gaussian elimination, then:

Note: the pivot positions are unique

free var's = variables with col's without pivots.

In general: it could be possible to choose other variables as free

Ex: $x + y = 4$

$y = 4 - x$
x free

$x = 4 - y$
y free

$(1 \; 1 \mid 4)$
Gauss: y free

② Homogeneous linear systems:

$$\left.\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= 0 \end{aligned}\right\}$$

general $m \times n$ homogeneous linear system;

$\boxed{b_1 = b_2 = \dots = b_m = 0}$

$rk(A) = rk(A|b)$ for all homogeneous sys.

$\Downarrow$

homogeneous sys. are <u>always consistent</u>

$x_1 = x_2 = \dots = x_n = 0$ is a solution

$(x_1, x_2, \dots, x_n) = (0, 0, \dots, 0)$ is called the <u>trivial solution</u>

homog. lin. sys.

- $rk(A) = n$: $(x_1, \dots, x_n) = (0, 0, \dots, 0)$ is the only solution; <u>one solution</u>
- $rk(A) < n$: there are <u>infinitely many solution</u> ($(x_1, \dots, x_n) = (0, \dots, 0)$ is one of them)

<u>Ex</u>:

$$\left.\begin{aligned} x + 2y - z &= 0 \\ 2x - y + 3z &= 0 \\ 3x + y + 2z &= 0 \end{aligned}\right\} \quad 3 \times 3 \text{ homog. lin. sys.}$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{-2,\,-3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 5 \\ 0 & -5 & 5 \end{pmatrix} \xrightarrow{-1} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

coeff. matrix echelon form

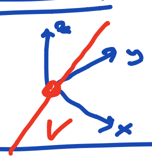
$$\left.\begin{aligned} rk(A) &= 2 \\ n &= 3 \end{aligned}\right\} \quad n - rk(A) = 3 - 2 = 1 \Rightarrow \text{1 free variable, infinitely many solutions}$$

$x + 2y - z = 0$
$-5y + 5z = 0$
~~$0 = 0$~~

<u>Solution</u>:

$(x, y, z) = (-z, z, z) = z \cdot (-1, 1, 1)$

with z free



$x + 2y - z = 0$

$x = -2y + z$
$= -2z + z$

$x = -z$

z is free

$-5y + 5z = 0$

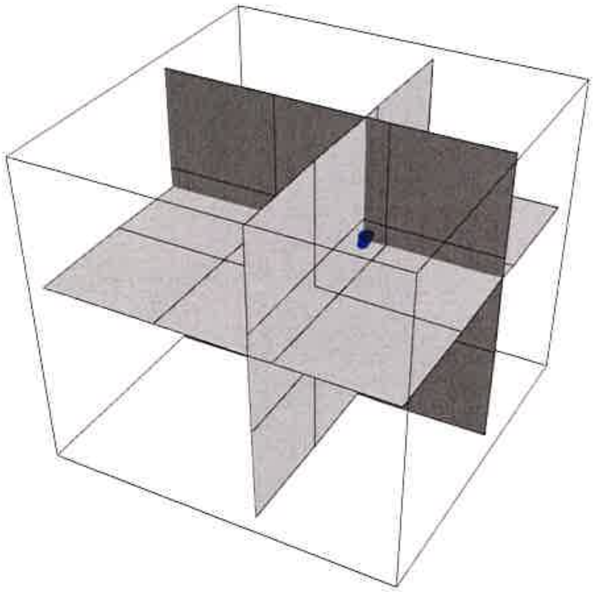
$-5y = -5z$
$\;\; \div(-5) \quad \div(-5)$

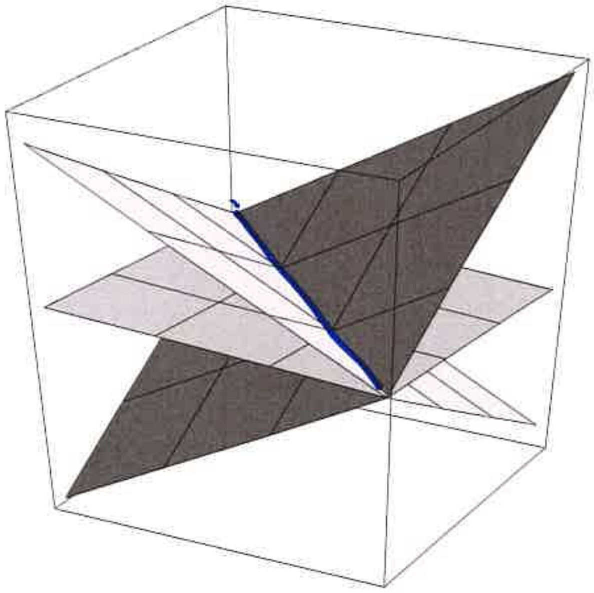
$y = z$

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

i) The planes intersect in one point. *(one solution)*

ii) The planes intersect in one line. *(infinitely many solutions)*





iii) There is not point in common to all three planes. *(no solution)*