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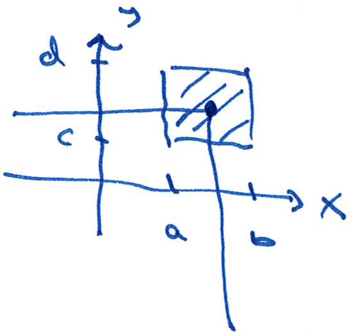
1 Repetisjon og oppgaveregning

Repetisjon: – sette opp og regne ut dobbeltintegral

– simultane fordelinger:



tetthetsfunksjon $f(x,y)$ slik at

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = p(a \le X \le b, c \le Y \le d)$$

kumulativ fordelingsfn: $F(a,b) = p(X \le a, Y \le b)$

$$= \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)\,dy\,dx$$

$$f(x,y) = F''_{xy}$$

marginalene: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

[I] Problem 3.10

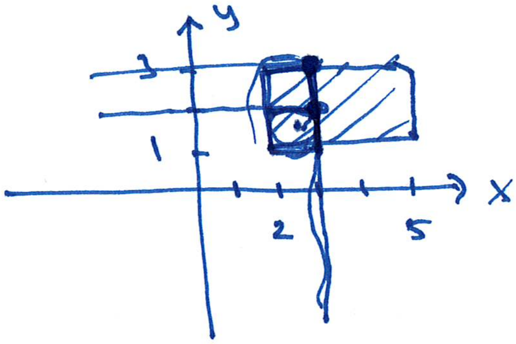
$$f(x,y) = \begin{cases} \frac{1}{91} x y^2, & 2 \le x \le 5,\ 1 \le y \le 3 \\ 0, & \text{ellers} \end{cases}$$

(a) $P(2 \le X \le 3, 1 \le Y \le 2)$

$= \underline{F(3,2)} = P(X \le 3, Y \le 2)$

(c) $P(2 \le X \le 3)$

$= \underline{F(3,3)}$

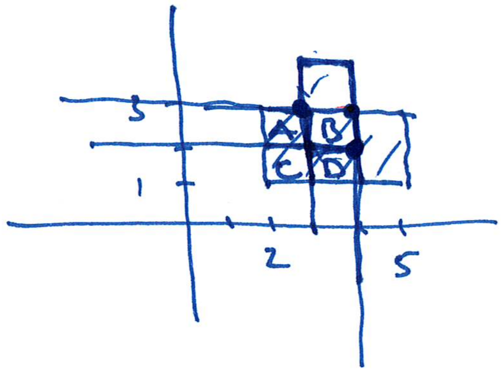


Kumulativ sannsynlighets-fordeling $F(x,y)$

$P(3 \le X \le 4, 2 \le Y \le 4)$

$= P(3 \le X \le 4, 2 \le Y \le 3)$ B

$= \underbrace{F(4,3)}_{A+B+C+D} - \underbrace{F(4,2)}_{C+D} - \underbrace{F(3,3)}_{A+C} + \underbrace{F(3,2)}_{C}$



② Forventning, varians og kovarians

X, Y simultant fordelte kont. stok. var med tetthetsfunksjon: $f(x,y)$

Marginaler:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Forventning:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy$$

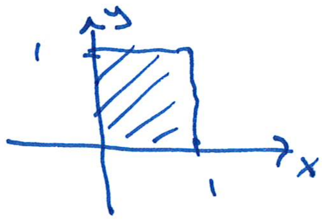
$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f(x,y)\,dy\,dx = \int_{-\infty}^{\infty} x \cdot \underbrace{\int_{-\infty}^{\infty} f(x,y)\,dy}_{f_X(x)}\,dx = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot f(x,y)\,dy\,dx = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy$$

Fakta: $h(X,Y)$ stokastisk variabel

$$E[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) \cdot f(x,y)\,dy\,dx$$

Eks: $f(x,y) = \begin{cases} x+y, & 0 \le x,y \le 1 \\ 0, & \text{ellers} \end{cases}$



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^1 x+y\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{1}{2}y^2\right]_0^1 dx = \int_0^1 x + \tfrac{1}{2}\, dx = \left[\frac{1}{2}x^2 + \frac{1}{2}x\right]_0^1 = \underline{1} \quad \text{ok}$$

$$E(X) = \int_0^1 x \cdot (x + \tfrac{1}{2})\, dx \qquad f_X(x) = \int_0^1 x + y \, dy$$

$$= \int_0^1 x^2 + \frac{1}{2}x \, dx = \left[\frac{1}{3}x^3 + \frac{1}{2}\cdot\frac{1}{2}x^2\right]_0^1 \qquad = \left[xy + \frac{1}{2}y^2\right]_0^1 = \underline{x + \frac{1}{2}}$$

for $0 \leq x \leq 1$

$$= (\tfrac{1}{3} + \tfrac{1}{4}) - 0 = \underline{\underline{7/12}}$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot f(x,y)\, dy\, dx = \int_0^1\int_0^1 y \cdot (x+y)\, dy\, dx$$

$$= \int_0^1\int_0^1 xy + y^2 \, dy\, dx = \int_0^1 \left[x \cdot \frac{1}{2}y^2 + \frac{1}{3}y^3\right]_0^1 dx$$

$$= \int_0^1 \frac{1}{2}x + \frac{1}{3} \, dx = \left[\frac{1}{2}\cdot\frac{1}{2}x^2 + \frac{1}{3}x\right]_0^1 = \frac{1}{4} + \frac{1}{3} - 0 = \underline{\underline{\frac{7}{12}}}$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f(x,y)\, dy\, dx = \int_0^1\int_0^1 xy(x+y)\, dy\, dx$$

$$= \int_0^1\int_0^1 x^2y + xy^2 \, dy\, dx = \int_0^1 \left[x^2 \cdot \frac{1}{2}y^2 + x \cdot \frac{1}{3}y^3\right]_0^1 dx$$

$$= \int_0^1 \frac{1}{2}x^2 + \frac{1}{3}x \, dx = \left[\frac{1}{2}\cdot\frac{1}{3}x^3 + \frac{1}{3}\cdot\frac{1}{2}x^2\right]_0^1 = \frac{1}{6} + \frac{1}{6} - 0$$

$$= 2/6 = 1/3 \;//$$

Varians og kovarians

$$Var(X) = E[(X-\mu_X)^2] = \underline{E(X^2) - E(X)^2} \qquad \mu_X = E(X)$$

$$Var(Y) = E[(Y-\mu_Y)^2] = \underline{E(Y^2) - E(Y)^2} \qquad \mu_Y = E(Y)$$

1 Eks:

$E(X) = 7/12$

$$E(X^2) = \int_0^1 x^2 \cdot (x + 1/2)\,dx = \int_0^1 x^3 + \tfrac{1}{2}x^2\,dx$$

$$= \left[\tfrac{1}{4}x^4 + \tfrac{1}{2}\cdot\tfrac{1}{3}x^3\right]_0^1 = 1/4 + 1/6 = 3/12 + 2/12 = \underline{5/12}$$

$$Var(X) = E(X^2) - E(X)^2 = 5/12 - (7/12)^2 = \frac{5 \cdot 12}{12 \cdot 12} - \frac{49}{12^2}$$

$$= \underline{\underline{\frac{11}{144}}} \qquad \sigma_X = \sqrt{\frac{11}{144}} = \underline{\underline{\sqrt{11}/12}}$$

Defn:

$$Cov(X,Y) = E[(X-\mu_X)\cdot(Y-\mu_Y)] = E[XY - \underline{\mu_X Y} - \underline{\mu_Y X} + \mu_X\mu_Y]$$

$$= E(XY) - \mu_X E(Y) - \cancel{\mu_Y E(X)} + \cancel{\mu_X \mu_Y}$$

$$\boxed{Cov(X,Y) = E(XY) - E(X)\cdot E(Y)}$$

Merk: $Cov(X,X) = Var(X)$ $\qquad Cov(X,Y) = Cov(Y,X)$

$Cov(Y,Y) = Var(Y)$

<u>Eks:</u> $E(X) = 7/12 \quad E(Y) = 7/12 \qquad E(XY) = 1/3$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$$

$$= \frac{1\cdot 4\cdot 12}{3\cdot 4\cdot 12} - \frac{7}{12}\cdot\frac{7}{12} = \frac{48-49}{144} = \underline{\underline{-\frac{1}{144}}}$$

$$\mathrm{Var}(X+Y) = E((X+Y)^2) - E(X+Y)^2$$

$$= E(X^2+2XY+Y^2) - (E(X)+E(Y))^2$$

$$= E(X^2) + 2E(XY) + E(Y^2) - (E(X)^2 + 2E(X)E(Y) + E(Y)^2)$$

$$= \underbrace{E(X^2) - E(X)^2}_{\mathrm{Var}(X)} + \underbrace{2E(XY) - 2E(X)E(Y)}_{2\,\mathrm{Cov}(X,Y)} + \underbrace{E(Y^2) - E(Y)^2}_{\mathrm{Var}(Y)}$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y)$$

$$\parallel$$

$$\mathrm{Cov}(X+Y, X+Y) = \mathrm{Cov}(X,X) + \mathrm{Cov}(Y,X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,Y)$$

$$= \underline{\mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y)}$$

<u>Eks:</u> $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y)$

$$= \frac{11}{144} + 2\cdot\left(-\frac{1}{144}\right) + \frac{11}{144} = \frac{20}{144} = \underline{\underline{\frac{5}{36}}}$$

X, Y simultant fordelte
a, b, c konst.
Z, Z_1, Z_2: uttrykk i X og Y

<u>Noen nyttige formler:</u>

i) $E(aX+bY+c) = E(aX)+E(bY)+E(c) = \underline{aE(X)+bE(Y)+c}$

ii) $Var(Z) = E(Z^2)-E(Z)^2 \geq 0$

iii) $Cov(Z_1, Z_2) = E(Z_1 \cdot Z_2) - E(Z_1)\cdot E(Z_2)$

iv) $Var(X+Y) = Var(X) + 2\,Cov(X,Y) + Var(Y)$

$$Var(aX+bY+c) = a^2\,Var(X) + 2ab\,Cov(X,Y) + b^2\,Var(Y)$$

<u>Sjekke:</u> $Var(Z+c) = E[(Z+c)^2] - E(Z+c)^2$

$$= E(Z^2+2cZ+c^2) - (E(Z)+c)^2$$

$$= \underline{E(Z^2)} + \cancel{2cE(Z)} + \underbrace{E(c^2)}_{=\cancel{c^2}} - (\underline{E(Z)^2} + \cancel{2cE(Z)} + \cancel{c^2})$$

$$= Var(Z)$$

$$Var(aX+bY) = Cov(\underline{a}X+bY, \underline{a}X+bY)$$

$$= Cov(aX,aX) + Cov(aX,bY) + Cov(bY,aX) + Cov(bY,bY)$$

$$= a^2\,Var(X) + \underbrace{ab\,Cov(X,Y) + ab\,Cov(X,Y)}_{2ab\cdot Cov(X,Y)} + b^2\,Var(Y)$$

Porteføljer:

A : Avkastning X
B : — " — Y

Porteføljens avkastning: $aX + bY$

Eks: $a = 1/2$, $b = 1/2$: $\frac{1}{2}X + \frac{1}{2}Y$

forventede avkastn. vektor

$$E(aX+bY) = a\,E(X) + b\cdot E(Y) = \begin{pmatrix} E(X) & E(Y) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$Var(aX+bY) = a^2 Var(X) + 2ab\, Cov(X,Y) + b^2 \cdot Var(Y) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

kovarians-matrisen

③ Uavhengighet:

Defn. To simultant fordelte stokastiske variabler X og Y er uavhengige hvis

$$\boxed{f(x,y) = f_X(x) \cdot f_Y(y)}$$

Fakta: Hvis X og Y er uavhengige, så er $\mathrm{Cov}(X,Y) = 0$.

Bevis:
$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) \cdot f_Y(y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x\, f_X(x) \underline{\int_{-\infty}^{\infty} y\, f_Y(y)\, dy}\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot E(Y)\, dx = E(Y) \cdot \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= E(Y) \cdot E(X) = E(X) \cdot E(Y) \Rightarrow \underline{\mathrm{Cov}(X,Y) = 0}$$

Merk: Selv om $\mathrm{Cov}(X,Y) = 0$, trenger ikke X og Y være uavhengige!

Eks: $f(x,y) = \begin{cases} x+y & , \ 0 \le x,y \le 1 \\ 0 & , \ \text{ellers} \end{cases}$

$\text{Cor}(X,Y) = -1/144 \neq 0 \Rightarrow X$ og Y ikke uavhengige

$$P(X \le 1/2, Y \le 1/2) = \int_0^{1/2}\int_0^{1/2} x+y \, dy \, dx$$

$$= \int_0^{1/2} \left[xy + \tfrac{1}{2}y^2\right]_0^{1/2} dx = \int_0^{1/2} \tfrac{1}{2}x + \tfrac{1}{8} \, dx = \left[\tfrac{1}{2}\cdot\tfrac{1}{2}x^2 + \tfrac{1}{8}x\right]_0^{1/2}$$

$$= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

$P(X \le 1/2)$

$\parallel$

$$\int_0^{1/2} x + 1/2 \, dx = \left[\tfrac{1}{2}x^2 + \tfrac{1}{2}x\right]_0^{1/2} = \tfrac{1}{2}\cdot\tfrac{1}{4} + \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{2}{2} = \underline{3/8}$$

Eks: $f(x,y) = \begin{cases} kxy & , \ 0 \le x,y \le 1 \\ 0 & , \ \text{ellers} \end{cases}$

$$f_X(x) = \int_0^1 kxy \, dy = \left[kx \cdot \tfrac{1}{2}y^2\right]_0^1 = \underline{\tfrac{1}{2}kx} , \ 0 \le x \le 1$$

$$\int_0^1 \tfrac{1}{2}kx \, dx = \left[\tfrac{1}{2}k \cdot \tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{4}k = 1 \Rightarrow \underline{k=4}$$

$\Rightarrow f(x,y) = 4xy , \ 0 \le x,y \le 1$

$f_X(x) = 2x , \ 0 \le x \le 1 \qquad f_Y(y) = \int_0^1 4xy \, dx$

$= \left[4y \cdot \tfrac{1}{2}x^2\right]_0^1 = \underline{2y} , \ 0 \le y \le 1$

X,Y uavhengige: $f(x,y) = f_X(x) \cdot f_Y(y)$

Sjekk: $f(x,y) = 4xy , \ 0 \le x,y \le 1$

$f_X(x) \cdot f_Y(y) = 2x \cdot 2y = 4xy , \ 0 \le x,y \le 1$ ok, X og Y uavhengige