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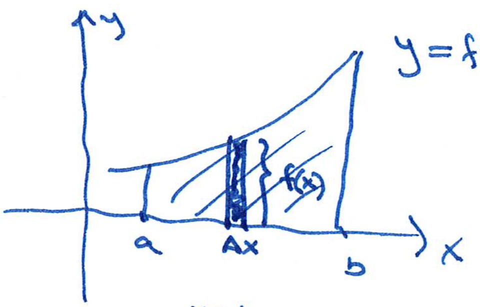
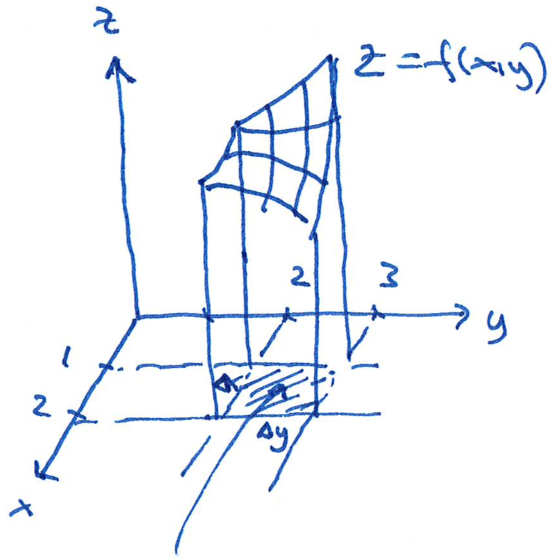
2 Dobbelt integral



Areal: $\int_{x=a}^{x=b} f(x)\,dx$

$\approx$

Summen av $f(x)\cdot\Delta x$



$\left\{\begin{matrix} 1 \le x \le 2 \\ 2 \le y \le 3 \end{matrix}\right\}$

Volumet under grafen til f når $1 \le x \le 2$ og $2 \le y \le 3$

$$\Delta V \cong \Delta x \cdot \Delta y \cdot f(x,y)$$

Volum: $\int_{x=1}^{x=2} \underbrace{\int_{2=y}^{3=y} f(x,y)\,dy}\,dx$

Eks: $f(x,y) = xy^2$ når $1 \le x \le 2$
$2 \le y \le 3$

$$\underbrace{\int_1^2 \int_2^3 xy^2 \, dy}_{\frac{19}{3}x} \, dx$$

$$\int_2^3 xy^2 \, dy$$

$$= x \cdot \int_2^3 y^2 \, dy$$

$$= x \cdot \left[\tfrac{1}{3} y^3\right]_2^3$$

$$= x \cdot \tfrac{1}{3} \cdot (3^3 - 2^3)$$

$$= \underline{\tfrac{19}{3} x}$$

$$= \int_1^2 \frac{19}{3} x \, dx = \frac{19}{3} \cdot \left[\frac{1}{2} x^2\right]_1^2$$

$$= \frac{19}{3} \cdot \frac{1}{2}(4-1) = \frac{19}{\cancel{6}_2} \cdot \cancel{3} = \underline{\underline{\frac{19}{2}}}$$

Faktum: Hvis $f(x,y) \ge 0$ for alle (x,y) slik at $a \le x \le b$ og $c \le y \le d$, så er volumet under grafen til f på det sikke rektangelet:

$$V = \int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy$$

3 Simultane fordelinger (kont. stok. variabler)

To stokastiske variabler X og Y som avhenger av utfallet i det samme stokastiske forsøket.

Eks: Trekker en tilfeldig valgt mnd i løpet av de siste 10 årene

X = mnds avkastn. til SP500

Y = — " — OSEBX

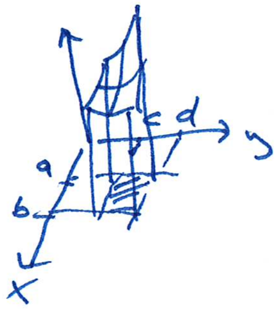
Simultan

Sannsynlighetstettheten til X og Y : $f(x,y)$

"og"

Egenskap: $p(a \le X \le b \,,\, c \le Y \le d)$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$



Krav: i) $f(x,y) \ge 0$ for alle x,y

ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

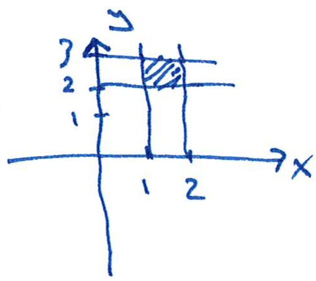
$f_{X,Y}(x,y) = f(x,y)$

Eks: $f(x,y) = \begin{cases} kxy^2 & , \ 1 \le x \le 2 \\ & \ \ \ 2 \le y \le 3 \\ 0 & , \ \text{ellers} \end{cases}$

Sjekk: i) ok hvis $k>0$ (eller $k=0$)

ii) $\int_1^2 \int_2^3 kxy^2\,dy\,dx = 1$ ok når $k = 2/19$

$$k \cdot \underbrace{\int_1^2 \int_2^3 xy^2\,dy\,dx}_{19/2} = 1 \quad \Big| \cdot \frac{2}{19}$$



$$k = 2/19$$

Konklusjon:

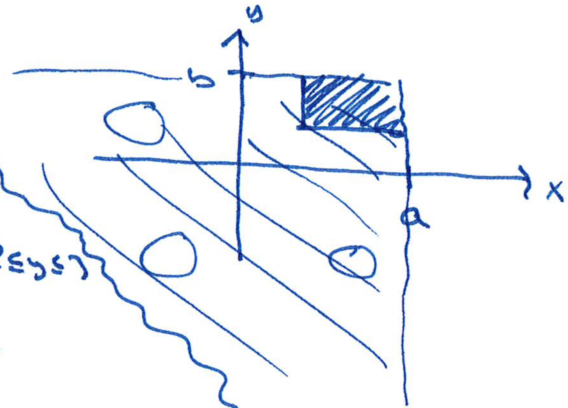
$$k = \frac{2}{19} \Rightarrow f(x,y) = \begin{cases} \frac{2}{19} xy^2 & , \ 1 \le x \le 2, 2 \le y \le 3 \\ 0 & , \ \text{ellers} \end{cases}$$

Kumulativ sannsynlighetsfordeling

X,Y simultant fordelte
stok. variabler
m/ sannsynlighetstetthet
$f(x,y)$

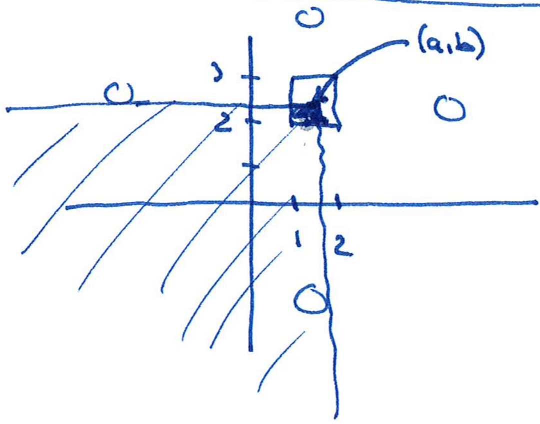
Defn

$$F(a,b) = p(X \le a, Y \le b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy\,dx$$



Eks: $f(x,y) = \begin{cases} \frac{2}{19} x y^2, & 1 \le x \le 2,\ 2 \le y \le 3 \\ 0, & \text{ellers} \end{cases}$

$F(a,b) = ?$



$1 \le a \le 2,\ 2 \le b \le 3$:

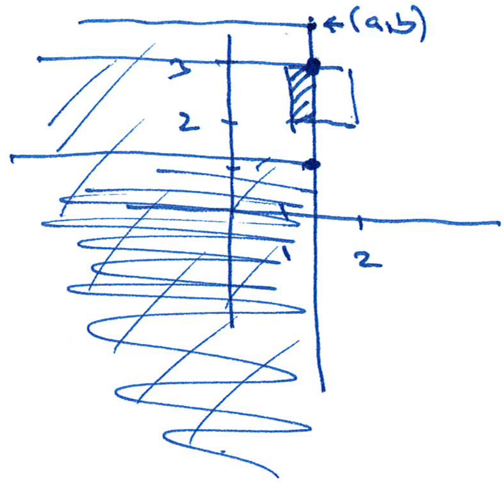
$$F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy\,dx$$

$$= \int_{1}^{a}\underbrace{\int_{2}^{b} \frac{2}{19} x y^2\,dy}\,dx$$

$$= \int_{1}^{a}\left[\frac{2}{19}x \cdot \frac{1}{3}y^3\right]_2^b dx = \int_{1}^{a} \frac{2}{19}x\left(\frac{1}{3}b^3 - \frac{1}{3}\cdot 2^3\right) dx$$

$$= \left(\frac{b^3}{3} - \frac{8}{3}\right)\cdot \frac{\cancel{2}}{19}\left[\frac{1}{\cancel{2}}x^2\right]_1^a = \frac{1}{19}\left(\frac{b^3}{3} - \frac{8}{3}\right)\cdot (a^2 - 1)$$

$$= \frac{1}{57}(a^2-1)(b^3-8) \quad , \quad 1 \le a \le 2,\ 2 \le b \le 3$$



$1 \le a \le 2,\ b > 3$:

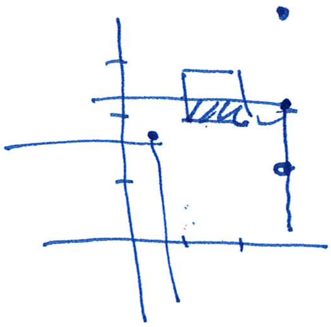
$$F(a,b) = F(a,3) = \frac{1}{57}(a^2-1)\cdot 19$$

$$= \frac{1}{3}(a^2-1) \quad , \quad 1 \le a \le 2,\ b > 3$$

$1 \le a \le 2,\ b < 2$:

$$F(a,b) = 0 \quad , \quad 1 \le a \le 2,\ b < 2$$

$a > 2$:



i) $a > 2,\ 2 \le b \le 3$:

$$F(a,b) = F(2,b) = \frac{1}{57}\cdot 3\cdot(b^3-8) = \frac{1}{19}(b^3-8)$$

ii) $a > 2,\ b > 3$: $F(a,b) = 1$

iii) $a > 2,\ b < 2$: $F(a,b) = 0$

Skrivemåte:

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx$$

$a < 1$: $F(a,b) = 0$

Konklusjon:

$$F(a,b) = \begin{cases} 1 , & a > 2,\ b > 3 \\ \frac{1}{57}(a^2-1)(b^3-8) , & 1 \le a \le 2,\ 2 \le b \le 3 \\ \frac{1}{3}(a^2-1) , & 1 \le a \le 2,\ b > 3 \\ \frac{1}{19}(b^3-8) , & a > 2,\ 2 \le b \le 3 \\ 0 , & \text{ellers} \end{cases}$$

<u>Fakta:</u> Hvis X, Y simultant fordelte med kumulativ fordeling $F(x,y)$, så er sannsynlighetstettheten

$$f(x,y) = F''_{xy} = F''_{yx} .$$

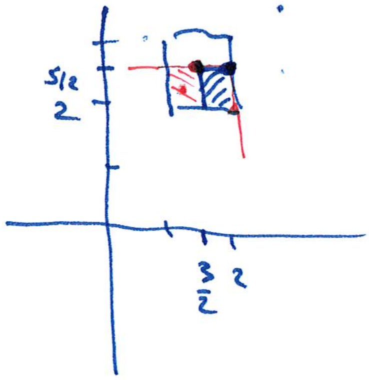
<u>Eks:</u> $F(x,y) = \frac{1}{57}(x^2-1)(y^3-8)$ når $1 \le x \le 2,\ 2 \le y \le 3$

$$\Rightarrow f(x,y) = F''_{xy} = \left(\frac{1}{57}(y^3-8)\cdot 2x\right)'_y = \frac{1}{57}\cdot 2x \cdot 3y^2$$

$$= \frac{2\cancel{6}xy^2}{\cancel{3}\cdot 19} = \frac{2xy^2}{19} = \frac{2}{19}xy^2 = f(x,y)$$

$$P(X \ge 3/2,\ Y \le 5/2)$$

$$= \int_{3/2}^{2}\int_{2}^{5/2} \frac{2}{19} x y^2 \, dy\, dx$$

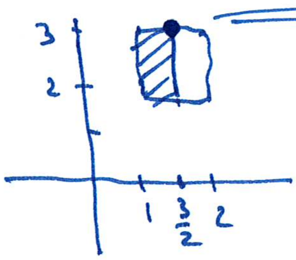


$$= F(2, 5/2) - F(3/2, 5/2)$$

<u>Alt:</u> F

$$= \frac{1}{57}(4-1)\left(\left(\tfrac{5}{2}\right)^3 - 8\right) - \frac{1}{57}\left(\left(\tfrac{3}{2}\right)^2 - 1\right)\left(\left(\tfrac{5}{2}\right)^3 - 8\right)$$

$$= \frac{1}{57}\left(\frac{125}{8} - \frac{64}{8}\right)\left(3 - \frac{9-4}{4}\right) = \frac{1}{57}\cdot\frac{61}{8}\cdot\frac{7}{4} = \frac{427}{57\cdot 32}$$

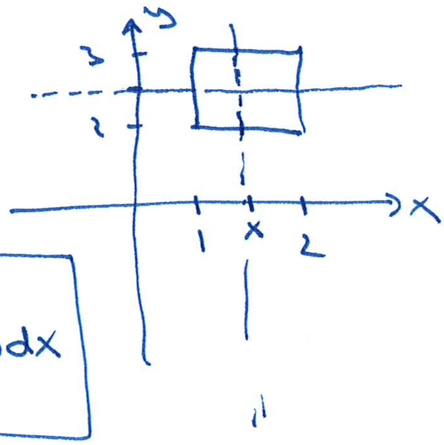


$$P(1 \le X \le 3/2) = F(3/2, 3)$$

$$= \frac{1}{57}\left(\frac{9}{4} - 1\right)(27-8)$$

$$= \frac{1}{3\,\cancel{57}}\cdot\frac{5}{4}\cdot\cancel{19} = \frac{5}{12}$$

Marginaler:



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx}$$

$$= \int_2^3 \frac{2}{19} x y^2\,dy = \frac{2}{19} x \left[\frac{1}{3} y^3\right]_2^3 = \frac{2}{19} x \left(\frac{1}{3}\cdot 27 - \frac{1}{3}\cdot 8\right)$$

$$= \frac{2}{\cancel{57}_{3}} \cdot \cancel{19}\, x = \underline{\underline{\frac{2}{3} x}}\ , \quad 1 \le x \le 2 \quad \text{og} \quad \underline{\underline{f_X(x) = 0}} \ \text{ellers}$$

$$P\left(1 \le x \le 3/2\right) = \int_1^{3/2} \frac{2}{3} x\,dx = \frac{\cancel{2}}{3} \left[\frac{1}{\cancel{2}} x^2\right]_1^{3/2}$$

!

$$= \frac{1}{3}\left(\left(\frac{3}{2}\right)^2 - 1^2\right) = \frac{1}{3}\left(\frac{9}{4} - \frac{4}{4}\right) = \frac{1}{3} \cdot \frac{5}{4} = \underline{\underline{\frac{5}{12}}}$$

$$\int_1^{3/2} \underbrace{\int_{-\infty}^{\infty} f(x,y)\,dy}_{= f_X(x)}\,dx = \int_1^{3/2} f_X(x)\,dx$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_1^2 x \cdot \frac{2}{3} x\,dx = \frac{2}{3}\left[\frac{1}{3} x^3\right]_1^2$$

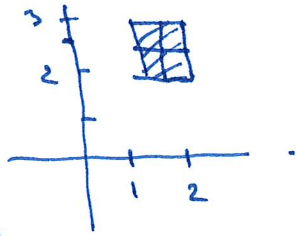
$$= \frac{2}{9}\left(2^3 - 1^3\right) = \underline{\underline{14/9}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_1^2 \frac{2}{19} x y^2\,dx = \frac{\cancel{2}}{19} y^2 \left[\frac{1}{\cancel{2}} x^2\right]_1^2$$

$$= \frac{1}{19} y^2 \left(2^2 - 1^2\right) = \underline{\frac{3}{19} y^2}\ , \quad 2 \le y \le 3$$

$$f_Y(y) = 0\ , \ \text{ellers}$$

$$f(x,y) = \begin{cases} \frac{2}{19} x y^2 & , \; 1 \le x \le 2, \; 2 \le y \le 3 \\ 0 & , \; \text{ellers} \end{cases}$$



$$\boxed{\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \frac{2}{3}x , & 1 \le x \le 2 \\ 0 , & \text{ellers} \end{cases} \\ f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx = \begin{cases} \frac{3}{19}y^2 , & 2 \le y \le 3 \\ 0 , & \text{ellers} \end{cases} \end{aligned}}$$

<u>marginalene</u>

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_1^2 x \cdot \frac{2}{3}x \, dx = \frac{14}{9}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_1^2 x^2 \cdot \frac{2}{3}x \, dx = \frac{\cancel{2}}{3}\,\frac{1}{\cancel{4}_{\,2}} x^4 \Big]_1^2 = \frac{1}{6}(16-1) = \frac{15}{6}$$

$$\begin{aligned} Var(X) = E(X^2) - E(X)^2 &= \frac{15}{6} - \left(\frac{14}{9}\right)^2 = \frac{15}{6} - \frac{14 \cdot 14}{9 \cdot 9} \\ &= \frac{15 \cdot 27}{6 \cdot 27} - \frac{14^2 \cdot 2}{81 \cdot 2} \\ &= \frac{405 - 392}{162} = \underline{\underline{13/162}} \end{aligned}$$