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5.1g er 5.1e) i "2 printing"

① Repetisjon

Ortogonal diagonalisering: $P^T A P = D$ hvor D er diagonal

A $n\times n$-matrise og $P^T = P^{-1}$ (P er ortogonal matrise)

$P = (\underline{v}_1 | \dots | \underline{v}_n)$ s.a. $\begin{cases} \underline{v}_i \cdot \underline{v}_j = 0 & i \neq j \\ \underline{v}_i \cdot \underline{v}_i = 1 \end{cases}$

ortogonal

Resultat: A ortogonal diagonaliserbar
$\Updownarrow$
A symmetrisk

Eks: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$: Egenverdier: $\lambda^2 - 2\lambda - 3 = 0$

[DA] 5.2

$\lambda = 3, \lambda = -1$ $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

"vanlig diagonalisering"

Egenvektorer: $\lambda = 3$: $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ $\lambda = -1$: $\begin{pmatrix} 2 & 2 \\ \cancel{2} & \cancel{2} \end{pmatrix}$

$-2x + 2y = 0$ $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$ $2x + 2y = 0$, y fri

y fri $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Ortogonal diagonalisering:

$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ $P = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$ $= \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\underline{v}_1 \cdot \underline{v}_2 = 1 \cdot (-1) + 1 \cdot 1 = 0$ ok

$\underline{v}_1 \cdot \underline{v}_1 = 2$ $\underline{v}_2 \cdot \underline{v}_2 = 2$

Faktum: Hvis A er symmetrisk og $\underline{v}_1, \underline{v}_2$ er egenvektorer for A med ulike egenverdier, da er $\underline{v}_1 \cdot \underline{v}_2 = 0$.

Kvadratiske former

$f(x_1, x_2, \ldots, x_n) = \; - - - \; = \underline{x}^T \cdot A \cdot \underline{x}$

polyn. uttrykk der alle ledd har grad 2

A symm $n \times n$-matrise

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Eks: $\left.\begin{array}{l} f(x,y,z) = x^2 + y^2 + z^2 \\ \quad + xy - xz + yz \end{array}\right\} = \underline{x}^T A \underline{x}$

$$A = \begin{array}{cc} \begin{pmatrix} 1 & 1/2 & -1/2 \\ 1/2 & 1 & 1/2 \\ -1/2 & 1/2 & 1 \end{pmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \\ \begin{array}{ccc} x & y & z \end{array} & \end{array}$$

Definitthet: f pos semidefn. $\Leftrightarrow$ $f(\underline{x}) \geq 0$ for alle $\underline{x}$

" neg " $f(\underline{x}) \leq 0$ —"—

Husk: $f(\underline{0}) = 0$

f indefn.  alle andre tilfeller

 f pos. defn. $\Leftrightarrow$ $f(\underline{x}) > 0$ for alle $\underline{x} \neq \underline{0}$

 " neg. " $f(\underline{x}) < 0$ —"—

Resultat:

$f(\underline{x}) = \underline{x}^T A \underline{x}$

A har egenverdier $\lambda_1, \lambda_2, \ldots, \lambda_n$

f pos. semidefn.	$\Leftrightarrow$	$\lambda_1, \ldots, \lambda_n \geq 0$
" neg "		$\lambda_1, \ldots, \lambda_n \leq 0$
" indefn.		alle andre tilfeller
f pos. defn.	$\Leftrightarrow$	$\lambda_1, \ldots, \lambda_n > 0$
" neg "	$\Leftrightarrow$	$\lambda_1, \ldots, \lambda_n < 0$

Oppgaver

[E] 4.12 b) $f(x,y,z) = x^2+y^2+z^2 + xy - xz + yz$

$$A = \begin{pmatrix} 1 & 1/2 & -1/2 \\ 1/2 & 1 & 1/2 \\ -1/2 & 1/2 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1/2 & -1/2 \\ 1/2 & 1-\lambda & 1/2 \\ -1/2 & 1/2 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot\left[(1-\lambda)^2 - 1/4\right] - \frac{1}{2}\left[\frac{1}{2}(1-\lambda) + 1/4\right] - \frac{1}{2}\left[1/4 + \frac{1}{2}(1-\lambda)\right] = 0$$

$4\lambda^2 - 8\lambda + 3 = 0$

$$\lambda = \frac{8 \pm \sqrt{(-8)^2 - 4\cdot 4\cdot 3}}{2\cdot 4} = \frac{8\pm\sqrt{16}}{8} = \frac{8\pm 4}{8} = 12/8,\ 4/8$$

$12/8 = 3/2$, $4/8 = 1/2$

$4(\lambda - 3/2)(\lambda - 1/2)$

$$(1-\lambda)\cdot(\lambda^2 - 2\lambda + 3/4) - 1\,(3/4 - \lambda/2) = 0 \quad |\cdot 4$$

$$(1-\lambda)\,(\underbrace{4\lambda^2 - 8\lambda + 3}) - (3 - 2\lambda) = 0$$

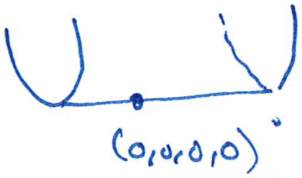
$$4(1-\lambda)(\lambda - 3/2)(\lambda - 1/2) + \underbrace{(2\lambda - 3)}_{=\,2(\lambda - 3/2)} = 0$$

$$(\lambda - 3/2)\cdot\left[4\cdot(1-\lambda)(\lambda - 1/2) + 2\right] = 0$$

$\underline{\lambda = 3/2}$ eller $-4\lambda^2 + 6\lambda = 0$

$\lambda(-4\lambda + 6) = 0$

$\underline{\lambda = 0}$ eller $\lambda = 6/4 = \underline{3/2}$



f pos. semidefn. siden $\lambda_1, \lambda_2, \lambda_3 \geq 0$

(men ikke pos. defn. siden $\lambda_2 = 0$)

$\Rightarrow$ minimumsverdien til f er $f_{min} = 0$

Alt: Multiplisere ut venstresiden

$$-4\lambda^3 + 12\lambda^2 - 9\lambda = 0$$

$$-\lambda\,(4\lambda^2 - 12\lambda + 9) = 0$$

$\underline{\lambda = 0}$, $\underline{\lambda = 3/2}$, $\underline{\lambda = 3/2}$

② Kvadratiske former og ortogonal diagonalisering

Eks: $f(x,y) = 17x^2 - 8xy + 23y^2$

$$= \underline{x}^T \underbrace{\begin{pmatrix} 17 & -4 \\ -4 & 23 \end{pmatrix}}_{A} \underline{x}$$

f pos. def.

Ortogonal diag. av A:

$D = \begin{pmatrix} 15 & 0 \\ 0 & 25 \end{pmatrix}$ $\quad P = \left(\underline{v_1} \,\vdots\, \underline{v_2} \right)$

$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

$P^T A P = D$

$\underline{x} = P\underline{u}$

$\begin{pmatrix} x \\ y \end{pmatrix} = P \cdot \begin{pmatrix} u \\ v \end{pmatrix} \mid P^{-1} \cdot$

$P^T \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

$\underline{u} = P^T \cdot \underline{x}$

$f(\underline{x}) = \underline{x}^T A \underline{x}$

$= (P\underline{u})^T A (P\underline{u})$

$= \underline{u}^T \underbrace{P^T \cdot A \cdot P}_{D} \underline{u}$

$= \underline{u}^T D \underline{u}$

$= (u\ v) \begin{pmatrix} 15 & 0 \\ 0 & 25 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$= 15u^2 + 25v^2$

$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$u = \frac{1}{\sqrt{5}}(2x + y)$

$v = \frac{1}{\sqrt{5}}(-x + 2y)$

Egenverdiene til A:

$\lambda^2 - 40\lambda + 17 \cdot 23 - 16 = 0$

$\lambda^2 - 40\lambda + 375 = 0$

$\lambda = 15, \ \lambda = 25$

Egenvektorene:

$\lambda = 15$: $\begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \Big] \cdot 2$

$\begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix}$

$2x - 4y = 0$, y fri

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = 25$: $\begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix} \Big] \cdot -\frac{1}{2}$

$\begin{pmatrix} -8 & -4 \\ 0 & 0 \end{pmatrix}$

$-8x - 4y = 0$, y fri

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y/2 \\ y \end{pmatrix} = y/2 \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\left\| \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

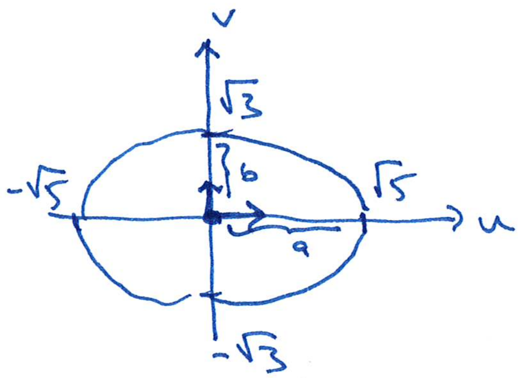
$\underline{v_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \underline{v_2} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$f(x,y) = 17x^2 - 8xy + 23y^2 = 75$$

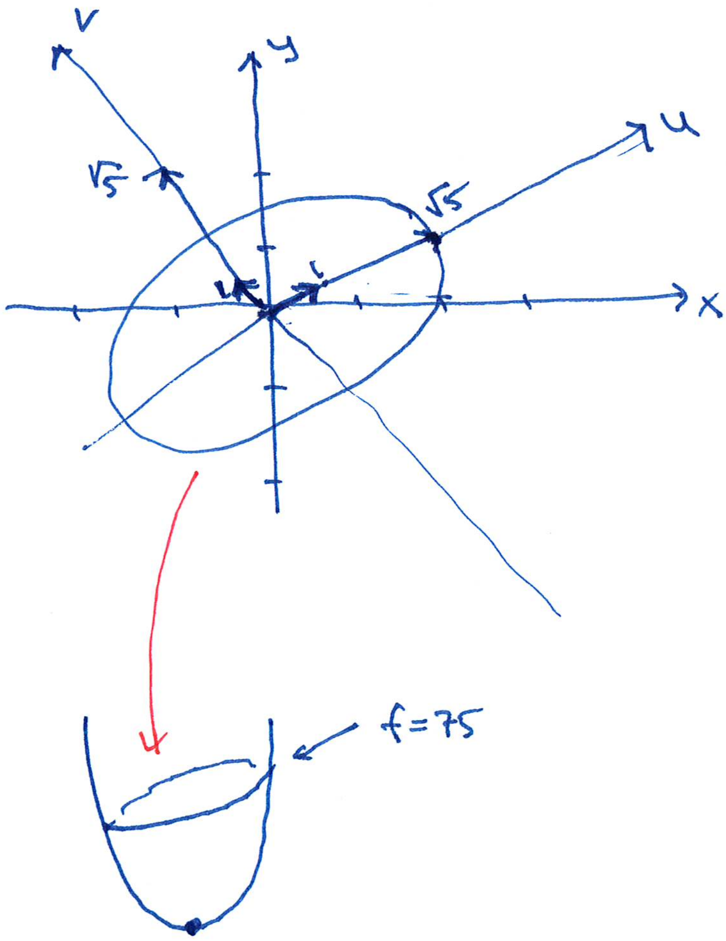
$$= \underbrace{15u^2 + 25v^2}_{\text{ellipse}} = 75 \quad | :75$$

$$\frac{15u^2}{75} + \frac{25v^2}{75} = 1$$

$$\frac{u^2}{5} + \frac{v^2}{3} = 1$$



ellipse w/ sentrum i $(0,0)$
og halvakser $a = \sqrt{5}$, $b = \sqrt{3}$



enhetsvektor
langs u-aksen

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

langs v-aksen

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(3) Optimering av kvadratiske funksjoner

Eks: $f(x,y) = \underbrace{x^2 - 4xy + 3y^2}_{\text{grad 2, kvadr. form}} + \underbrace{7x - 8y}_{\text{grad 1, lineær form}} + \underbrace{5}_{\text{grad 0, konst.}}$

$\| \qquad\qquad$

$\underline{x}^T A \underline{x}$, med $\qquad B\underline{x}$, med $\qquad C$

$A = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \qquad B = (7 \ \ -8)$

$B\underline{x} = (7 \ \ -8)\begin{pmatrix} x \\ y \end{pmatrix} = 7x - 8y$

Generelt: $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$ kvadr. funksjon i var. $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Optimering: max/min $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$

(1) Stasjonære pkt: $f'_{x_1} = 0,\ f'_{x_2} = 0,\ \ldots,\ f'_{x_n} = 0$

Eks: $f(x,y) = x^2 - 4xy + 3y^2 + 7x - 8y + 5$

$f'_x = 2x - 4y + 7 = 0 \qquad 2x - 4y = -7$

$f'_y = -4x + 6y - 8 = 0 \qquad -4x + 6y = 8$

$\left(\begin{array}{cc|c} 2 & -4 & -7 \\ -4 & 6 & 8 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & -4 & -7 \\ 0 & \boxed{-2} & -6 \end{array}\right)$

$2x - 4y = -7$

$-2y = -6$

Stasj. pkt: $x = 5/2$, $y = 3$

$(x,y) = (5/2, 3)$

På matriseform $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$, der

$A = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$, $B = (7 \;\; -8)$ $C = 5$

$$f'(\underline{x}) = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = 2A\underline{x} + B^T + \underline{0} = 2 \cdot \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 7 \\ -8 \end{pmatrix}$$

$$= 2 \cdot \begin{pmatrix} x - 2y \\ -2x + 3y \end{pmatrix} + \begin{pmatrix} 7 \\ -8 \end{pmatrix}$$

$$= \begin{pmatrix} 2x - 4y + 7 \\ -4x + 6y - 8 \end{pmatrix} \leftarrow \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$$

Fakta:

i) $f(\underline{x}) = \underline{x}^T A \underline{x} \Rightarrow f'(\underline{x}) = 2A\underline{x}$

ii) $f(\underline{x}) = B\underline{x} \Rightarrow f'(\underline{x}) = B^T$

iii) $f(\underline{x}) = C \Rightarrow f'(\underline{x}) = \underline{0}$

Merk: Hvis $B = 0$, så er $\underline{x} = \underline{0}$ et stasjonært pkt.

$$f'(\underline{x}) = 2A\underline{x} + B^T = \underline{0}$$

$2A\underline{x} = -B^T$ lineært system

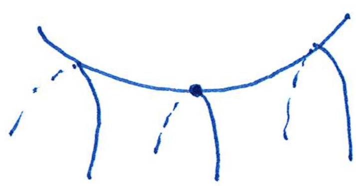
$A\underline{x} = -\frac{1}{2}B^T$

Fakta: $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$

i) f konveks $\Leftrightarrow$ A positiv semidefn. $\Rightarrow$ stasjonære pkt er globale min. pkt.

ii) f konkav $\Leftrightarrow$ A negativ semidefn. $\Rightarrow$ stasjonære pkt er globale maks. pkt.

iii) A indefn. $\Rightarrow$ stasjonære pkt er sadelpkt



<u>Eks:</u> $f(\underline{x}) = \underline{x}^T \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \underline{x} + (7 \ \ -8)\underline{x} + 5$

$$f'(\underline{x}) = 2\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}\underline{x} + \begin{pmatrix} 7 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5/2 \\ 3 \end{pmatrix}$$

$$\underline{(x,y) = (5/2, 3)}$$

$A = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$: Egenverdier

$$\lambda^2 - 4\lambda - 1 = 0$$

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm \sqrt{4} \cdot \sqrt{5}}{2} = 2 \pm \sqrt{5}$$

$$\left.\begin{aligned} \lambda_1 &= 2 + \sqrt{5} > 0 \\ \lambda_2 &= 2 - \sqrt{5} < 0 \end{aligned}\right\} A \text{ \underline{indef.}}$$

$\Rightarrow (x,y) = (\underline{5/2, 3})$ er et sadelpkt.

<u>Konkl:</u> f har hverken maks eller min.

<u>Alt:</u> $A = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$

$$\text{tr} A = 1 + 3 = 4 = \lambda_1 + \lambda_2$$

$$\det A = 3 - 4 = -1 = \lambda_1 \cdot \lambda_2 \Rightarrow \underline{\text{indef.}}$$