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Oppgaver fra læreboken	[E] 4.11ac, 4.12, 4.13ab, 4.15 - 4.17

① Repetisjon

A $n \times n$-matrise

Egenverdi / egenvektor for A

$$A \cdot \underline{v} = \lambda \underline{v}$$
$$\Updownarrow$$
$$(*) \quad (A - \lambda I)\underline{v} = \underline{0}$$

egenverdi λ s.a. $(*)$ har ikke trivielle løsn $\underline{v} \neq \underline{0}$

egenvektor $\underline{v} \neq \underline{0}$ som er løsn. av $(*)$ for en bestemt λ

egenrom $E_\lambda = \{\underline{v} : A\underline{v} = \lambda \underline{v}\}$

Metode:

i) λ egenverdi $\Leftrightarrow |A - \lambda I| = 0$ (polyn. likn. av grad n)

ii) $\underline{v}$ i E_λ: løsn. av $(A - \lambda I)\underline{v} = \underline{0}$

Gauss! → (lin. homogent system)

Fakta: Hvis A har n egenverdier $\lambda_1, \lambda_2, \dots, \lambda_n$ så har vi:

$$\boxed{\begin{aligned} \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n &= |A| \\ \lambda_1 + \lambda_2 + \ldots + \lambda_n &= tr(A) \end{aligned}}$$

← Summen av tallene på diagonalen

Eks:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1^2 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda^2 - tr(A)\lambda + |A| = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 1, \lambda = 3$$

A er diagonaliserbar hvis $P^{-1}AP = D$ er diagonal for en invertibel matrise P.

A diagonaliserbar $\Leftrightarrow$ det fins n lineært uavhengige egenvektorer $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ for A med $A\underline{v_1} = \lambda_1 \cdot \underline{v_1}$, $A\underline{v_2} = \lambda_2 \underline{v_2}$, ..., $A\underline{v_n} = \lambda_n \cdot \underline{v_n}$

I så fall:

$$P = \left(\underline{v_1} \,\vdots\, \underline{v_2} \,\vdots\, \ldots \,\vdots\, \underline{v_n} \right)$$

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Oppg. [DA] 4.5:

$$A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & 5 & 0 \\ 1 & 1 & 4 \end{pmatrix}$$

Egenverdier:

$$\begin{vmatrix} 1-\lambda & 7 & -2 \\ 0 & 5-\lambda & 0 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$+(5-\lambda) \cdot \begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(5-\lambda) \cdot (\lambda^2 - 5\lambda + 6) = 0$$

$$\lambda_1 = 5 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

$s=2$: $\lambda = 2$ eller $\lambda = 3$

E_2:

$$\begin{pmatrix} -1 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} -1 & 7 & -2 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} -x + 7y - 2z &= 0 \\ 8y &= 0 \end{aligned} \qquad \begin{aligned} -x &= 2z \quad x = -2z \\ y &= 0 \end{aligned}$$

z fri

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} -2t \\ 0 \\ t \end{pmatrix} = t \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Basis for E_2: $\underline{v_1} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

E_3: $\begin{pmatrix} -2 & 7 & -2 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -2 & 7 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 9 & 0 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x + y + z = 0 \quad x = -z$

$-y = 0 \quad y = 0$

z fri

Basis for E_3: $\underline{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = t \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Egenvektorer for A når $s = 2$: $\quad \underline{v_1} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Kun to lin. uavh. egenvektorer for A $\Rightarrow$ A ikke diag.

Fakta

Hvis A er $n \times n$-matrise med n forskjellige egenverdier $\lambda_1, \lambda_2, \ldots, \lambda_n$ så er A diagonaliserbar.

(2) Ortogonal diagonalisering

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ n-vektorer

Defn Mengden $\{\underline{v_1}, \ldots, \underline{v_n}\}$ kalles ortogonal hvis $\underline{v_i} \perp \underline{v_j}$ når $i \neq j \Leftrightarrow$ (i) holder

Mengden $\{\underline{v_1}, \ldots, \underline{v_n}\}$ kalles ortonormal hvis det i tillegg er slik at $\|\underline{v_i}\| = 1 \Leftrightarrow$ (i) og (ii) holder

(i) $\underline{v_i} \cdot \underline{v_j} = 0$ for alle $i \neq j$

(ii) $\underline{v_i} \cdot \underline{v_i} = 1$ for alle i

Defn: En $n \times n$-matrise P kalles <u>ortogonal</u> hvis
$$P^T = P^{-1}.$$

Defn:

A $n \times n$-matrise — En <u>ortogonal diagonalisering</u> av A er en diagonalisering $P^{-1}AP = D$ slik at P er <u>ortogonal</u>.

$$P^T A P = D$$

Fakta: $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ (n-vektorer) $\rightsquigarrow$ $P = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$ ($n \times n$-matrise)

$$\boxed{\{\underline{v_1}, \dots, \underline{v_n}\} \text{ er en ortonormal mengde} \iff P \text{ er ortogonal matrise } (P^T = P^{-1})}$$

Forklaring:

$$P^T \cdot P = I \iff P^T = P^{-1}$$
$$\Updownarrow$$
$$\underline{v_1} \cdot \underline{v_1} = 1,\ \underline{v_2} \cdot \underline{v_2} = 1, \dots$$
og
$$\underline{v_1} \cdot \underline{v_2} = 0,\ \underline{v_1} \cdot \underline{v_3} = 0, \dots$$
$$\Updownarrow$$
ortonormal mengde

$$P^T \cdot P = \begin{pmatrix} \text{---}\ \underline{v_1}\ \text{---} \\ \text{---}\ \underline{v_2}\ \text{---} \\ \vdots \\ \text{---}\ \underline{v_n}\ \text{---} \end{pmatrix} \cdot \begin{pmatrix} \underline{v_1} & \underline{v_2} & \dots & \underline{v_n} \end{pmatrix} = \begin{pmatrix} \underline{v_1} \cdot \underline{v_1} & \underline{v_1} \cdot \underline{v_2} & \dots & \underline{v_1} \cdot \underline{v_n} \\ \underline{v_2} \cdot \underline{v_1} & \underline{v_2} \cdot \underline{v_2} & \dots & \\ \vdots & \vdots & \ddots & \end{pmatrix}$$

Eks:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Egenverdier

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\underline{\lambda = 1}, \ \underline{\lambda = 3}$$

Egenvektorer:

$\underline{E_1}$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $x+y=0$, y fri

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

basis for E_1

$\underline{E_3}$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ $-x+y=0$, y fri

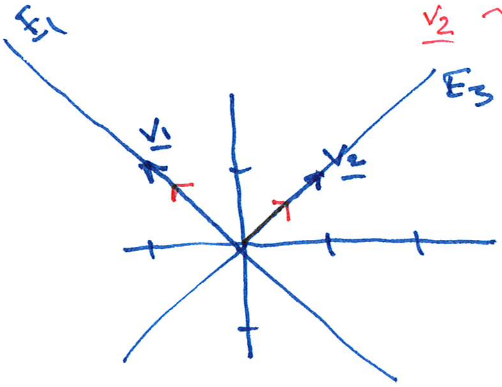
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

basis for E_3

$\underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to lin. uavh. egenvektorer

Normalisering: $\underline{v_1} \rightsquigarrow \frac{1}{\|v_1\|} \cdot \underline{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{v_2} \rightsquigarrow \frac{1}{\|v_2\|} \cdot \underline{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ 1 \end{pmatrix}$



Diagonalisering:

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{-2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\boxed{P^{-1}AP = D}$$

Er P ortogonal?

$$\underline{v_1} \cdot \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1 \cdot 1 + 1 \cdot 1 = 0$$

$$\underline{v_1} \cdot \underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2 \neq 1$$

$$\underline{v_2} \cdot \underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \neq 1$$

Ny P:

$$P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = -\frac{1}{2} + \frac{1}{2} = 0$$

$$\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{2} + \frac{1}{2} = 1$$

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{2} + \frac{1}{2} = 1$$

ortogonal diagonalisering:

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{-1} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \qquad = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = P^T$$

Fakta: (1) Hvis A er symmetrisk, og $\underline{v_1}$ og v_2 er to egenvektorer med ulike egenverdier, så er $\underline{v_1}\cdot\underline{v_2}=0$.

(2) A er ortogonal diagonaliserbar $\Leftrightarrow$ A symmetrisk.

"Bevis" av (1): $\underline{v_1}^T A \underline{v_2} = \underline{v_1}^T(\lambda_2 \underline{v_2}) = \lambda_2 \cdot \underline{v_1}^T \underline{v_2} = \lambda_2(\underline{v_1}\cdot\underline{v_2})$

$$(\underline{v_1}^T A \underline{v_2})^T = \underline{v_2}^T A^T \underline{v_1} = \underline{v_2}^T (A\underline{v_1}) = \underline{v_2}^T(\lambda_1 \underline{v_1})$$

$$= \lambda_1 (\underline{v_2}\cdot\underline{v_1}) = \lambda_1(\underline{v_1}\cdot\underline{v_2})$$

1x1-matrise

$$\Rightarrow \lambda_2(\underline{v_1}\cdot\underline{v_2}) = \lambda_1(\underline{v_1}\cdot\underline{v_2})$$

$$\underbrace{(\lambda_2-\lambda_1)}_{\neq 0}\cdot(\underline{v_1}\cdot\underline{v_2}) = 0 \Rightarrow \underline{v_1}\cdot\underline{v_2}=0$$

3 Kvadratiske former

Defn. En funksjon $f(x_1, x_2, \ldots, x_n)$ i n variabler kalles en kvadratisk form hvis f er et polynom der alle ledd har grad to.

Eks: $f(x,y) = x^2 + 4xy + 2y^2$

$f(x,y,z) = x^2 + 6xy + 2xz + 2y^2 - 4yz + 5z^2$

(kvadratiske former)

Fakta: Enhver kvadratisk form i n variabler kan skrives på matriseform som $\underline{x}^T A \underline{x}$ der

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{og } A \text{ er } n\times n\text{-matrise.}$$

Vi kan velge A symmetrisk, og da er A entydig.

Eks: $f(x,y) = x^2 + 4xy + 2y^2 \qquad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\| $$

$$(x \;\; y)\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} \qquad \underline{A = \begin{array}{cc|c} x & y & \\ \hline \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} & & \begin{array}{c} x \\ y \end{array} \end{array}}$$

$$\| $$

$$(x \;\; y)\cdot\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \left(\underline{x(ax+by)} + y(cx+dy)\right)$$

$$= \left(ax^2 + bxy + cxy + dy^2\right)$$

$$= \left(ax^2 + (b+c)xy + dy^2\right)$$

$$a = 1 \qquad b + c = 4 \qquad d = 2$$

$$(b = c = 2)$$

Eks: $f(x,y,z) = \underline{x^2} + \underline{6xy} + 2\underline{xz} + \underline{2y^2} - \underline{4yz} + \underline{5z^2}$

$$= \underline{x}^T A \underline{x} \qquad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$A = \begin{array}{c|c} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & -2 \\ 1 & -2 & 5 \end{pmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \\ \begin{array}{ccc} x & y & z \end{array} & \end{array}$$

er den symmetriske matrisen til den kvadr. formen.

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Definitthet til en kvadratisk form

$f(\underline{x}) = \underline{x}^T A \underline{x}$ kvadratisk form i n variabler på matriseform ved symm. matrise A.

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$

Defn: Den kvadratiske formen f (eller matrisen A) kalles

i) positiv semidefinit hvis $f(\underline{x}) \geq 0$ for alle $\underline{x}$

ii) negativ semidefinit " $f(\underline{x}) \leq 0$ —"—

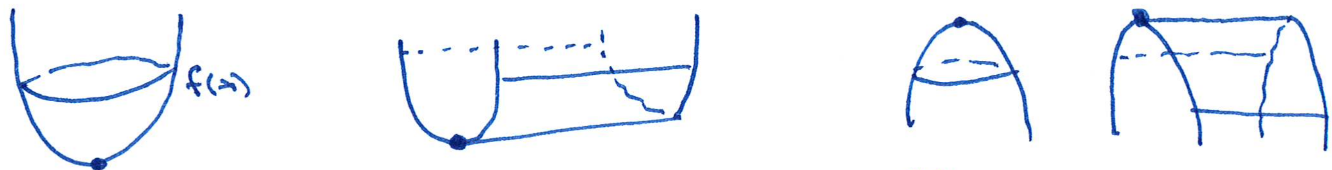
iii) indefinit ellers

Merk: $f(\underline{0}) = 0$

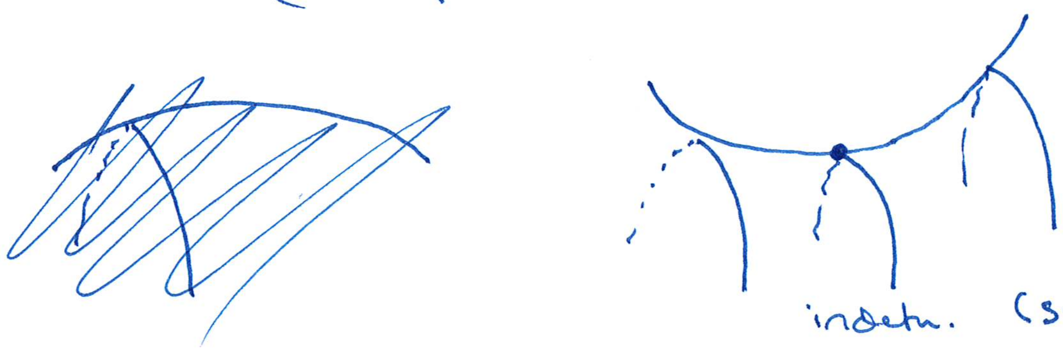
og dessuten:

positiv definit hvis $f(\underline{x}) > 0$ for alle $\underline{x} \neq \underline{0}$

negativ definit " $f(\underline{x}) < 0$ —"—



positiv defn. (min pkt) positiv semidefn. neg. defn. neg. semidefn. (maks pkt)



indefn. (sadelpkt)

Eks: $f(x,y) = x^2 + 4xy + 2y^2$ $\qquad f(1,0) = 1 > 0$

$= (x+2y)^2 - 2y^2 \qquad f(2,-1) = -2 < 0$

indefinit

Resultat:

Hvis $f(\underline{x}) = \underline{x}^T A \underline{x}$ er en kvadr. fkn i n variabler med symm. matrise A, med egenverdier $\lambda_1, \lambda_2, \dots, \lambda_n$.

Da har vi:

i) f pos. semidefn $\Leftrightarrow \lambda_1, \lambda_2, \dots, \lambda_n \geq 0$

ii) f neg. –"– $\Leftrightarrow \lambda_1, \lambda_2, \dots, \lambda_n \leq 0$

iii) f indefn. $\Leftrightarrow$ det fins både positive og negative egenverdier

Dessuten:

f pos. defn. $\Leftrightarrow \lambda_1, \dots, \lambda_n > 0$

f neg " $\Leftrightarrow \lambda_1, \dots, \lambda_n < 0$

Eks: $x^2 + 4xy + 2y^2 \rightsquigarrow A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \rightsquigarrow \lambda^2 - 3\lambda + (-2) = 0$

$\lambda = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$

indefn. $\Leftarrow$ $\lambda_1 = \frac{3+\sqrt{17}}{2} > 0$

$\lambda_2 = \frac{3-\sqrt{17}}{2} < 0$