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## 1 Repetisjon og oppgavegjennomgang

**Repetisjon:** – regning m/ matriser, spesielt matrisemultiplikasjon og transponering

– determinant

A $n\times n$-matrise $\rightsquigarrow$ $\det(A) = |A|$, et tall

Kofaktor-utvikling

V.h.a Gauss-eliminasjon

Regneregler: $\boxed{\begin{array}{l} |AB| = |A|\cdot|B| \\ |A^T| = |A| \end{array}}$

– inverse matriser

A $n\times n$-matrise

$A^{-1}$ er en matrise s.a. $A\cdot A^{-1} = I$

$A^{-1}\cdot A = I$

$|A| \neq 0$

$\Updownarrow$

$A^{-1}$ fins

(A invertibel)

$|A| \neq 0$: $A^{-1} = \frac{1}{|A|}\cdot \text{adj}(A)$

$= \frac{1}{|A|}\cdot \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}^T$

Gauss ?

**Oppgaver:**

[E] 3.9 c):

$$\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & -2 & 4 & 0 \\ 1 & -4 & 1 & -1 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 4 \\ 1 & -4 & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -4 & 3 \\ 0 & -6 & 0 \end{vmatrix} = +1(-4 \cdot 0 - 3 \cdot (-6)) \qquad \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & 4 \end{vmatrix} = +1(4+3)$$

$$= -1(18) - 1(7) = \cancel{44}\ -25 \neq 0$$

$\Rightarrow$ $\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\}$ lin. uavhengige

**Resultat:** $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ n-vektorer $\rightsquigarrow A = (\underline{v_1} \mid \underline{v_2} \mid \ldots \mid \underline{v_n})$

$\{\underline{v_1}, \ldots, \underline{v_n}\}$ lineært uavhengige
$\Updownarrow$
$|A| \neq 0$
$\Updownarrow$
$A^{-1}$ fins ($A$ inverterbar)
$\Updownarrow$
$A \cdot \underline{x} = \underline{b}$ har eksakt én løsn. for alle vektorer $\underline{b}$
$(\underline{x} = A^{-1} \cdot \underline{b})$

$\{\underline{v_1}, \ldots, \underline{v_n}\}$ lineært avh.
$\Updownarrow$
$|A| = 0$
$\Updownarrow$
$A^{-1}$ fins ikke
$\Updownarrow$
$A\underline{x} = \underline{0}$ har uendelig mange løsn.

$A\underline{x} = \underline{b}$ har ingen eller uendelig mange løsn. for alle $\underline{b}$

[DA] 3.11.

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} = -2 \cdot \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & +2 \end{vmatrix}$$

$$= -2\left(+3 \cdot \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}\right) = -6\left(4 \cdot 1 - 5 \cdot 1\right) = \underline{\underline{6}}$$

$4(2 \cdot 2 - (-3)(-1))$ = $4 \cdot 1$

$-5(3 \cdot 2 - (-1)(-5))$ = $-5 \cdot 1$

3.17 $|A| \neq 0$ : $A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A) = \frac{1}{|A|}(C_{ij})^T$

$|A| = 0$ : $A^{-1}$ fins ikke

Alt:

A $n \times n$-matrise $\quad (A|I) \to \dots \to (B|C)$

$$\boxed{\begin{array}{ll} \text{Hvis } B = I: & A^{-1} = C \\ \text{" } B \neq I: & A^{-1} \text{ fins ikke} \end{array}}$$

redusert trappeform = trappeform slik at:

(i) alle pivoter = 1

(ii) alle tall over en pivot = 0

d) $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ : $|A| = 3 \cdot 1 \cdot 2 = 6 \neq 0$ $\quad A^{-1} = \frac{1}{6}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}^T$

$$(A|I) = \left(\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \cdot 1/3 \\ \\ \cdot 1/2 \end{array} \to \left(\begin{array}{ccc|ccc} 1 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \begin{array}{l} \\ -1/3 \\ \\ \end{array}$$

trappeform

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \rightsquigarrow A^{-1} = \underline{\underline{\begin{pmatrix} 1/3 & -1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}}}$$

## ② Egenverdier og egenvektorer

$A$ $n \times n$-matrise

<u>Defn</u> En <u>egenverdi</u> for $A$ er et tall $\lambda$ slik at likningen

$$\boxed{A \cdot \underline{x} = \lambda \underline{x}} \quad (*)$$

har minst én løsn $\underline{x} \neq \underline{0}$ (dvs likningen har ikke-trivielle løsn.)

En <u>egenvektor</u> for $A$ med egenverdi $\lambda$ er en ikke-triviell løsn $\underline{x}$ av $(*)$.

<u>Eks:</u> $A = \begin{pmatrix} 1 & 5 \\ -2 & -6 \end{pmatrix} \rightsquigarrow \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\lambda$ tall

$$A\underline{x} = \begin{pmatrix} 1 & 5 \\ -2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 5y \\ -2x - 6y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \underline{x}$$

$$(*) \quad \begin{aligned} x + 5y &= \lambda x \\ -2x - 6y &= \lambda y \end{aligned} \iff \begin{aligned} x - \lambda x + 5y &= 0 \\ -2x - 6y - \lambda y &= 0 \end{aligned}$$

$$\Updownarrow$$

$$(A - \lambda I)\underline{x} = \underline{0} \iff \begin{pmatrix} 1-\lambda & 5 \\ -2 & -6-\lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{aligned} (1-\lambda)x + 5y &= 0 \\ -2x + (-6-\lambda)y &= 0 \end{aligned}$$

<u>Fakta:</u> $A\underline{x} = \lambda \underline{x}$ kan skrives på matriseform som $\boxed{(A - \lambda I)\underline{x} = \underline{0}}$

lineært system, homogent, m/parameter $\lambda$

$$\Updownarrow$$

$$A\underline{x} - \lambda \underline{x} = \underline{0}$$

$$A\underline{x} - \lambda I \underline{x} = \underline{0}$$

$$(A - \lambda I)\underline{x} = \underline{0}$$

$\lambda$ egenverdi for $A$ $\overset{\text{defn.}}{\Leftrightarrow}$ (*) $A\underline{x} = \lambda \underline{x}$ har ikke-trivielle løsn. $\Leftrightarrow$ $(A-\lambda I)\underline{x} = \underline{0}$ har ikke-trivielle løsn.

$\Updownarrow$

$|A-\lambda I| = 0$

Karakteristisk ligning

Eks:

$$A = \begin{pmatrix} 1 & 5 \\ -2 & -6 \end{pmatrix} : \quad \begin{vmatrix} 1-\lambda & 5 \\ -2 & -6-\lambda \end{vmatrix} = 0 \leftarrow \text{karakteristisk likn.}$$

$$(1-\lambda)\cdot(-6-\lambda) - 5(-2) = 0$$
$$-6 + 6\lambda - \lambda + \lambda^2 + 10 = 0$$
$$\boxed{\lambda^2 + 5\lambda + 4 = 0}$$
$$(\lambda+1)(\lambda+4) = 0$$
$$\underline{\lambda = -1} \text{ eller } \underline{\lambda = -4}$$

Viètes regel:

$x^2 - ax + b = 0$ har løsn. $x_1$ og $x_2$ hvis $x_1 + x_2 = a$, $x_1 \cdot x_2 = b$

Egenverdiene til $A$: $\lambda_1 = \underline{-1}$, $\lambda_2 = \underline{-4}$

Egenvektorer for $A$:

$\underline{\lambda = -1}$:
$$\begin{pmatrix} 2 & 5 \\ -2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gauss:
$$\begin{pmatrix} 2 & 5 \\ -2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 5 \\ 0 & 0 \end{pmatrix}$$

$2x + 5y = 0$, $y$ fri

$x = -5y/2$, $y$ fri ($y = t$)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5t/2 \\ t \end{pmatrix} = t \cdot \begin{pmatrix} -5/2 \\ 1 \end{pmatrix}$$

$E_{-1}$ = egenrommet for $\lambda = -1$
= samlingen av alle egenvektor for $\lambda = -1$

$\underline{v_1} = \begin{pmatrix} -5/2 \\ 1 \end{pmatrix}$: $E_{-1} = \text{span}(\underline{v_1})$

$\{\underline{v_1}\}$ basis for $E_{-1}$.

$\underline{\lambda = -4}$: $\begin{pmatrix} 5 & 5 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad \begin{pmatrix} 5 & 5 \\ -2 & -2 \end{pmatrix} \xrightarrow{2/5} \begin{pmatrix} 5 & 5 \\ 0 & 0 \end{pmatrix}$

$x = -y$ $\qquad 5x + 5y = 0$

$y$ fri $\qquad y$ fri

$(y = t)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = \underline{t \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$E_{-4} = \text{span}(\underline{v_2})$, $\underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

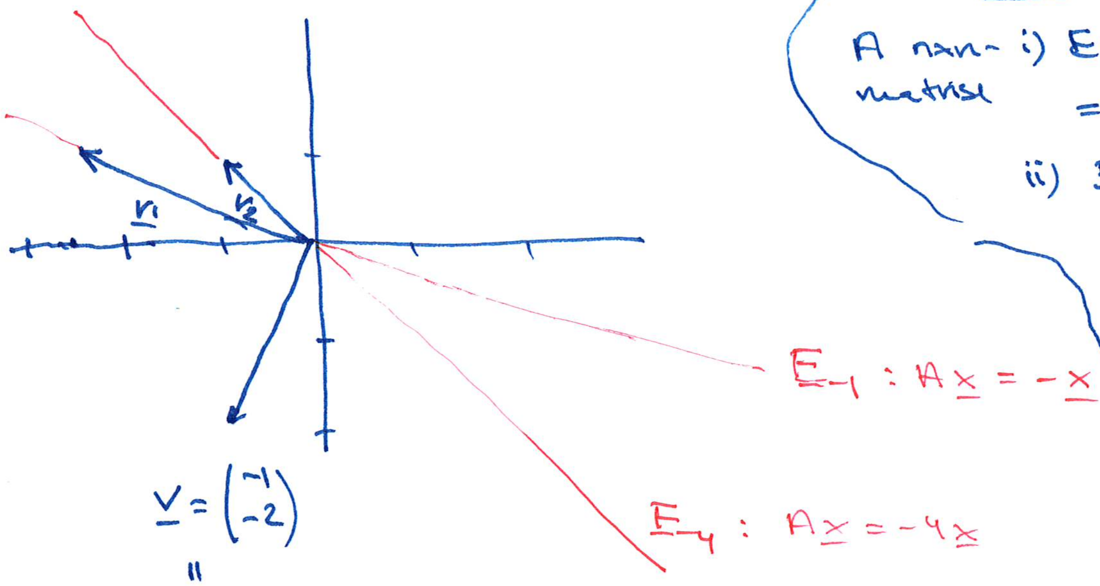
$\{\underline{v_2}\}$ basis for $E_{-4}$

**Oppsummering:**

$A$ $n \times n$-matrise

i) Egenverdier for $A$ $= $ løsn. av $|A - \lambda I| = 0$

ii) $E_\lambda$ = egenvektoren til $A$ m/ egenv. $\lambda$ er løsn. av $(A - \lambda I)\underline{x} = 0$ som vi kan finne vha Gauss-eliminasjon.




$\underline{v} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$\shortparallel$

$c_1 \cdot \underline{v_1} + c_2 \underline{v_2}$ ?

$$c_1 \cdot \begin{pmatrix} -5/2 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}:$$

$$\left(\begin{array}{cc|c} -5/2 & -1 & -1 \\ 1 & 1 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & -2 \\ -5/2 & -1 & -1 \end{array}\right) \cdot 2$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & -2 \\ -5 & -2 & -2 \end{array}\right) \xrightarrow{5} \left(\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 3 & -12 \end{array}\right)$$

$c_1 + c_2 = -2$
$3c_2 = -12$

$c_1 = 2$
$c_2 = -4$

$\underline{\text{Ja}}$: $\underline{v} = 2\underline{v_1} - 4\underline{v_2}$

$$A\underline{v} = A(2\underline{v_1} - 4\underline{v_2})$$
$$= 2A\underline{v_1} - 4A\underline{v_2}$$
$$= 2(-\underline{v_1}) - 4(-4\underline{v_2}) = -2\underline{v_1} + 16\underline{v_2}$$
$$= -2\begin{pmatrix} -5/2 \\ 1 \end{pmatrix} + 16\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -11 \\ 14 \end{pmatrix}$$

**Fakta:** $A$ $n\times n$-matrise $\rightsquigarrow$ $|A-\lambda I|=0$ er en polynomlikn. av grad $n$

kar. likn.

**Fakta:** $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ : $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=0$

tilfellet $n=2$

$(a-\lambda)(d-\lambda)-bc=0$

$ad-a\lambda-d\lambda+\lambda^2-bc=0$

$$\lambda^2-\underbrace{(a+d)}_{\substack{= \\ tr(A) \\ \text{(sporet til } A)}}\lambda+\underbrace{(ad-bc)}_{\substack{= \\ det(A) \\ \text{(det. til } A)}}=0$$

$\downarrow$

$$\boxed{\lambda^2-tr(A)\lambda+det(A)=0}$$

**Fakta:** Ikke alle matriser har egenverdier.

**Eks:** $A=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Kar. likn:

$\lambda^2-0\cdot\lambda+1=0$

$\lambda^2=-1$

ingen egenverdier (blant reelle tall)

$A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Kar. likn:

$\lambda^2-2\lambda+1=0$

$(\lambda-1)^2=0$

$\lambda_1=1$ eller $\lambda_2=1$

## ③ Diagonalisering

A $n \times n$-matrise

**Defn:** A er diagonaliserbar hvis det fins en invertibel matrise P slik at

$$\boxed{P^{-1}AP = D}$$

er en diagonal matrise D.

**Merk:** P: $P^{-1}AP = D$

$$PP^{-1}AP = PD$$
$$I\,AP = PD$$
$$\boxed{AP = PD}$$

$P = (\underline{v_1} \,\vdots\, \underline{v_2} \,\vdots\, \cdots \,\vdots\, \underline{v_n})$ $n \times n$-matrise

slik at $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ er lin. uavh.

og

$D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{pmatrix}$ er diagonal

$$AP = A \cdot (\underline{v_1} \,\vdots\, \underline{v_2} \,\vdots\, \cdots \,\vdots\, \underline{v_n}) = (A\underline{v_1} \,\vdots\, A\underline{v_2} \,\vdots\, \cdots \,\vdots\, A\underline{v_n})$$

$$\parallel \qquad\qquad\qquad\qquad\qquad\qquad \parallel$$

$$PD = (\underline{v_1} \,\vdots\, \underline{v_2} \,\vdots\, \cdots \,\vdots\, \underline{v_n}) \cdot \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{pmatrix} = (d_1\underline{v_1} \,\vdots\, d_2\underline{v_2} \,\vdots\, \cdots \,\vdots\, d_n\underline{v_n})$$

**Dette betyr:**

$$\left.\begin{aligned} A\underline{v_1} &= d_1\underline{v_1} \\ A\underline{v_2} &= d_2\underline{v_2} \\ &\vdots \\ A\underline{v_n} &= d_n\underline{v_n} \end{aligned}\right\}$$

$\underline{v_1}$ egenvektor m/ $\lambda = d_1$

$\underline{v_2}$ —"— m/ $\lambda = d_2$

$\vdots$

$\underline{v_n}$ —"— m/ $\lambda = d_n$

**Resultat:** $A$ $n \times n$-matrise

$A$ er diagonaliserbar $\Leftrightarrow$ $A$ har $n$ lineært uavhengige egenvektorer $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ med egenverdier $\lambda_1, \lambda_2, \ldots, \lambda_n$

I så fall:

$$P = \left( \underline{v}_1 \,|\, \underline{v}_2 \,|\, \ldots \,|\, \underline{v}_n \right) \quad \text{og} \quad D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

**Merk:** Hvis $A$ er en $n \times n$-matrise med $n$ forskjellige egenverdier (alltid i dette kurset), så er $A$ diagonaliserbar.

**Eks:** $A = \begin{pmatrix} 1 & 5 \\ -2 & -6 \end{pmatrix}$ $\quad \lambda_1 = -1 \quad$ Basis for $E_{-1}$: $\underline{v}_1 = \begin{pmatrix} -5/2 \\ 1 \end{pmatrix}$

$\lambda_2 = -4 \quad$ Basis for $E_{-4}$: $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Downarrow$

$D = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \quad P = \begin{pmatrix} -5/2 & -1 \\ 1 & 1 \end{pmatrix}$ gir $P^{-1}AP = D$

diagonalisering