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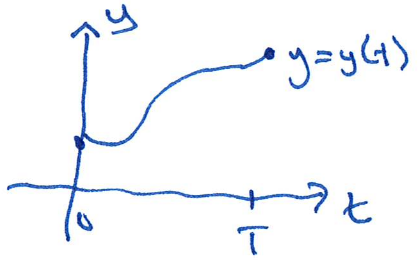
① Repetisjon

Variasjonsproblem: max/min $\underbrace{\int_0^T F(t,y,y')\,dt}_{= J(y) \text{ funksjonal } J}$ når $y(0)=y_0$, $y(T)=y_T$




Løsningsmetode:

i) Euler-likningen: $F'_y - \frac{d}{dt}\left(F'_{y'}\right) = 0$

Løsn. av Euler-likn. som også oppfyller $\left\{ \begin{array}{l} y(0)=y_0 \\ y(T)=y_T \end{array} \right\}$

= kandidater for max/min.

ii) For å avgjøre om kandidatfn. er max/min:

$H(F) = \begin{pmatrix} F''_{yy} & F''_{yy'} \\ F''_{y'y} & F''_{y'y'} \end{pmatrix}$ $= \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ : Hvis $H(F)$ er pos. semidefn. $(AC-B^2 \geq 0,\ A,C \geq 0)$ $\Rightarrow$ min

Hvis $H(F)$ er neg. semidefn. $(AC-B^2 \geq 0,\ A,C \leq 0)$ $\Rightarrow$ max

Oppg. 14.8 i [DA]

$$\max \int_0^T e^{-t/4} \ln(2K-K')\,dt \quad \text{når} \quad \begin{cases} K(0)=K_0 \\ K(T)=K_T \end{cases}$$

Euler-likn: $F(t,K,K') = \ln(2K-K')\cdot e^{-t/4}$

$y=K$
$y'=K'$

$$F'_K = \frac{1}{2K-K'}\cdot 2\cdot e^{-t/4} = e^{-t/4}\cdot \frac{2}{2K-K'}\,\frac{\cdot(2K-K')}{\cdot(2K-K')}$$

$$F'_{K'} = e^{-t/4}\cdot\frac{1}{2K-K'}\cdot(-1) = e^{-t/4}\cdot\frac{-1}{2K-K'}$$

$$\frac{d}{dt}(F'_{K'}) = \frac{d}{dt}\left(e^{-t/4}\cdot\frac{-1}{2K-K'}\right) = e^{-t/4}\cdot(-1/4)\cdot\frac{-1}{2K-K'}$$

$$+\,e^{-t/4}\cdot(-1)\left[(2K-K')^{-1}\right]'$$

$$= \frac{\frac{1}{4}\cdot e^{-t/4}}{2K-K'} - e^{-t/4}\cdot\left(-1\cdot(2K-K')^{-2}\cdot(2K'-K'')\right)$$

$$= \frac{e^{-t/4}}{(2K-K')^2}\cdot\left[\frac{1}{4}(2K-K') + 2K'-K''\right]$$

Euler: $F'_K - \frac{d}{dt}(F'_{K'}) = 0$

$$\frac{e^{-t/4}}{(2K-K')^2}\cdot\left[2(2K-K') - \left(\frac{1}{4}(2K-K') + 2K'-K''\right)\right] = 0$$

$$\frac{225}{16} - \frac{14\cdot 16}{16} = \frac{1}{16}$$

$$4K - 2K' - \frac{1}{2}K + \frac{1}{4}K' - 2K' + K'' = 0$$

$$K'' - \frac{15}{4}K' + \frac{7}{2}K = 0 \quad \text{Euler-likn.}$$

Kar. likn: $r^2 - \frac{15}{4}r + \frac{7}{2} = 0 \Rightarrow r = \dfrac{15/4 \pm\sqrt{(15/4)^2 - 4\cdot 7/2}}{2}$

$$r = \frac{15/4 \pm 1/4}{2} = 2,\ 14/8 = 7/4$$

Generell løsn.: $\underline{K = C_1 \cdot e^{2t} + C_2 e^{7/4\,t}}$

Finn $C_1$ og $C_2$ ved hjelp av $K(0) = K_0$, $K(T) = K_T$.

$\Rightarrow$ Kandidat for max.

**Er dette max?**

$$H(F) = \begin{pmatrix} F''_{KK} & F''_{K,K'} \\ F''_{K,K'} & F''_{K',K'} \end{pmatrix}$$

$$\frac{1}{2K-K'} = (2K-K')^{-1}$$

$$F'_K = e^{-t/4} \cdot \frac{2}{2K-K'}$$

$$F'_{K'} = e^{-t/4} \cdot \frac{-1}{2K-K'}$$

$$F''_{KK} = 2e^{-t/4} \cdot (-1) \cdot (2K-K')^{-2} \cdot 2 = \frac{e^{-t/4}}{(2K-K')^2} \cdot (-4)$$

$$F''_{K,K'} = 2e^{-t/4} \cdot (-1) \cdot (2K-K')^{-2} \cdot (-1) = \frac{e^{-t/4}}{(2K-K')^2} \cdot 2$$

$$F''_{K',K'} = \frac{-1 \cdot e^{-t/4}}{(2K-K')^2} \cdot (-1) \cdot (-1) = \frac{e^{-t/4}}{(2K-K')^2} \cdot (-1)$$

$$H(F) = \underbrace{\frac{e^{-t/4}}{(2K-K')^2}}_{\text{pos.}} \cdot \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

$AC - B^2 = 0$

$A = -4,\ C = -1$

F konkav $\Leftarrow$ neg. semidefin

$\Rightarrow$ Kandidatfn. $\underline{K(t) = C_1 e^{2t} + C_2 e^{7/4\,t}}$

løser variasjonsproblemet, dvs gir max i integralet.

## 2 Optimal kontrollteori

**Problem:** $\max/\min \int_0^T F(t,y,u)\,dt$ når $\begin{cases} y' = G(t,u,y) \\ y(0) = y_0 \\ y(T) = y_T \end{cases}$

$y$: tilstandsvariabel

$u$: kontrollvariabel

**Eks:** $\min \int_0^1 \underbrace{y^2+u^2}_{F}\,dt$ når $\begin{cases} y' = \overbrace{u}^{G} \\ y(0) = 0 \\ y(1) = 5 \end{cases}$ kontroll-problem

$= \min \int_0^1 y^2 + y'^2\,dt$ når $\begin{cases} y(0)=0 \\ y(1)=5 \end{cases}$ var. problem.

**Løsn:** Var. problem

$F = y^2 + y'^2$

$F'_y = 2y$

$F'_{y'} = 2y' \qquad \frac{d}{dt}(2y') = 2y''$

**Euler:**

$2y - 2y'' = 0 \quad | : (-2)$

$\underline{y'' - y = 0}$

Kar. likn: $r^2 - 1 = 0$

$r = \pm 1$

$y = \underline{C_1 \cdot e^t + C_2 e^{-t}}$

**min?**

$H(F) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \lambda_1 = 2, \ \lambda_2 = 2$

pos. semidef.

$\Downarrow$

$F$ konveks i $(y, y')$ $\nearrow$ **min**

**Kontrollproblem:** max/min $\int_0^T F(t,y,u)\,dt$ når $\begin{cases} y' = G(t,y,u) \\ y(0) = y_0 \\ y(T) = y_T \end{cases}$

**Pontryagins maksimumsprinsipp:**

$H = F(t,y,u) + p(t)\cdot G(t,y,u)$ Hamilton-funksjonen til problemet.

**Betingelser:**

$H'_u = 0$ og $H'_y = -p'(t)$

**normal løsn:** løsn. $y(t)$ av de to betingelsene i Pontryagins maks. prinsipp + start/slutt-betingelse.

**Eks:** min $\int_0^1 y^2 + u^2\,dt$ når $\begin{cases} y' = u \\ y(0) = 0 \\ y(1) = 5 \end{cases}$

$H = \underbrace{y^2+u^2}_{F} + p\cdot \underbrace{u}_{G}$

(1) $H'_u = 2u + p = 0$

(2) $H'_y = 2y = -p'(t)$

$y' = u$

$\Downarrow$

$\underline{u = y'}$: $2y' + p = 0$

$2y = -p'(t)$

$\underline{p = -2y'}$: $2y = -(-2y'')$

$2y = 2y''$

$0 = 2y'' - 2y \quad |:2$

$\underline{y'' - y = 0}$ $\quad r = \pm 1$

$y = C_1 e^t + C_2 e^{-t}$

Løsningsmetode: Kontrollproblem

i) Finn den normale løsn. vha. Pontragins maksimumsprinsipp: $H'_u = 0$

$H'_y = -p'$ $\boxed{H = F + p \cdot G}$

$y' = G(t, u, y)$

$+$

Kandidatfunksjon for max / min i integralet. $\leftarrow$ $y(0) = y_0$

$y(T) = y_T$

ii) Sjekk om kandidatfun. er max / min:

$$\begin{pmatrix} H''_{yy} & H''_{yu} \\ H''_{uy} & H''_{uu} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

pos. semidefin. $\Rightarrow$ min (H konveks i $(y, u)$)

neg. semidef. $\Rightarrow$ max (H konkav i $(y, u)$)

1 eks: $y = C_1 e^t + C_2 e^{-t}$

$H = y^2 + u^2 + p \cdot u$

$$\begin{pmatrix} H''_{yy} = 2 & H''_{yu} = 0 \\ H''_{uy} = 0 & H''_{uu} = 2 \end{pmatrix}$$

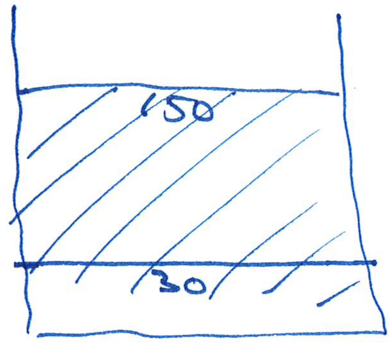
$\lambda_1 = 2$ pos. semidefin.

$\lambda_2 = 2$ $\Downarrow$

$\underline{y = C_1 e^t + C_2 e^{-t}}$

gir min. i integralet.

Eks:

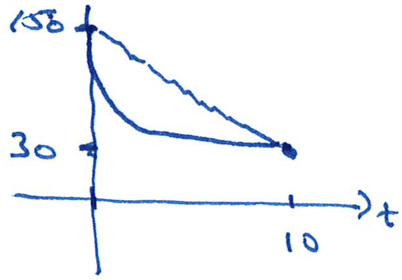



oljereservoar

$y(t)$ : antall enheter olje i reservoaret ved tid $t$

$y(0) = 150$

$y(10) = 30$




$u = -y'$

$$\max \int_0^{10} u(t) \cdot p_B(t) \cdot e^{-rt}\, dt \qquad \begin{cases} y' = -u \\ y(0) = 150 \\ y(10) = 30 \end{cases}$$

(maks. nåverdi av salgsverdien til oljen)

Eks: $\max \int_0^1 y - u^2 \, dt$ når $\begin{cases} y' = y + u \\ y(0) = 1 \\ y(1) = e - \frac{1}{2e} + \frac{1}{2} \end{cases}$

Pontryagins maks-prinsipp:

$$H = y - u^2 + p \cdot (y + u)$$

$\left.\begin{matrix} y = y(t) \\ u = u(t) \\ p = p(t) \end{matrix}\right\}$ vil finne optimale verdier for disse

Sjekk om $H$ er konkav i $(y,u)$:

$H'_y = 1 + p$

$H'_u = \underline{-2u + p}$

$$\begin{pmatrix} H''_{yy} = 0 & H''_{yu} = 0 \\ H''_{uy} = 0 & H''_{uu} = -2 \end{pmatrix}$$

$\lambda_1 = 0$, $\lambda_2 = -2$ neg. semidef.

$AC - B^2 = 0 \geq 0$ ✓

$A = 0,\ C = -2 \leq 0$ ✓

$\Rightarrow H$ $\underline{\underline{\text{konkav}}}$ i $(y,u)$

Kandidat fn. = normal løsn

$H'_u = -2u + p = 0$

$H'_y = 1 + p = -p'$

$y' = y + u$

$y(0) = 1$

$y(1) = e - \frac{1}{2e} + \frac{1}{2}$

$\frac{2u}{2} = \frac{p}{2} \Rightarrow u = p/2$

$p = 2u$

$1 + p = -p'$

$1 + 2u = -(2u)' = -2 \cdot u'$

$\boxed{2u' + 2u = -1} \quad :2$

$u' + u = -1/2 \leftarrow$

Alt: $y'' = y' + u' = y' - \frac{1}{2} - u$

$u = p/2 \Rightarrow u' = p'/2$

$= -(1+p)/2 = -1/2 - u$

$u = y' - y: \; y'' = \cancel{y'} - \frac{1}{2} - (\cancel{y'} - y)$

$y'' - y = -1/2$

Løser:

$u = u_h + u_p = \underline{Ce^{-t} - 1/2}$

$u_h$: $u' + u = 0$

$r + 1 = 0$

$r = -1$

$u_h = C \cdot e^{-t}$

$u_p$: $u' + u = -1/2$

Prøver $u = A$ (konst.)

$u' = 0$

$0 + A = -1/2 \quad A = -1/2$

$u_p = -1/2$

$y' = y + u$

$y' - y = Ce^{-t} - 1/2 \;|\cdot e^{-t}$

$(y \cdot e^{-t})' = C \cdot e^{-2t} - 1/2\, e^{-t}$

$y \cdot e^{-t} = \int Ce^{-2t} - 1/2\, e^{-t}\, dt = C \cdot \frac{e^{-2t}}{-2} - \frac{1}{2} \cdot \frac{e^{-t}}{-1} + D$

$ye^{-t} = \frac{C}{-2} e^{-2t} + \frac{1}{2} e^{-t} + D \;|\cdot e^{t}$

$y = -\frac{1}{2} C e^{-t} + \frac{1}{2} + D \cdot e^{t} = \underline{C_1 e^{-t} + C_2 e^{t} + 1/2}$

Int. faktor:

$a(t) = -1 \quad \int -1\, dt = -t + c$

int faktor: $v = e^{-t}$

$y(0)=1$ $\quad 1 = c_1 + c_2 + 1/2$

$y(1) = e - \frac{1}{2e} + 1/2$ $\quad e - \frac{1}{2e} + \cancel{1/2} = c_1 e^{-1} + c_2 \cdot e + \cancel{1/2}$

$\Rightarrow c_1 + c_2 = 1/2 \quad \boxed{c_2 = 1/2 - c_1}$

$e - \frac{1}{2e} = \frac{c_1}{e} + (1/2 - c_1)\cdot e \quad | \cdot e$

$e^2 - \frac{1}{2} = c_1 + (1/2 - c_1) e^2$

$\frac{\frac{1}{2}(e^2-1)}{1-e^2} = \frac{1}{2}e^2 - \frac{1}{2} = e^2 - \frac{1}{2} - \frac{1}{2}e^2 = \frac{c_1 \cdot (1-e^2)}{1 \cdot e^2}$

$-1/2 = c_1$

$c_1 = -1/2 \quad c_2 = 1/2 - (-1/2) = 1$

$y^* = -\frac{1}{2}e^{-t} + e^t + 1/2$

gir max i kontrollproblemet

$u = y' - y$

**Alt 3:** $\max \int_0^1 y - u^2 \, dt$ når $\begin{cases} y' = y + u \\ y(0) = 1 \\ y(1) = e - \frac{1}{2e} + 1/2 \end{cases}$

$= \max \int_0^1 y - (y'-y)^2 \, dt$ når $\begin{cases} y(0) = 1 \\ y(1) = e - \frac{1}{2e} + 1/2 \end{cases}$

Var. problem:

$F = y - (y'-y)^2 = y - (y'^2 - 2yy' + y^2)$

$= y - y^2 + 2yy' - y'^2$

$F'_y = 1 - 2y + 2y'$

$F'_{y'} = 2y - 2y' \quad \frac{d}{dt}(2y - 2y') = 2y' - 2y''$

**Euler:** $1 - 2y + \cancel{2y'} - (\cancel{2y'} - 2y'') = 0$

$2y'' - 2y = -1 \quad | : 2 \longrightarrow y'' - y = -1/2$