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## 1 Introduksjon til kurset

Tema: lineær algebra Forelesn. 1 - 6
sannsynlighetsregn. Forelesn. 7 - 10
differensial. likn. Forelesn. 11 - 15
og optimal kontrollteori

Anvendelser:

* Finans min $\underbrace{\text{Varians}}_{\text{andregradsfunksjon}}$ gitt $\underbrace{\text{forventet avkastn.} = r}_{\text{lineær bibetingelse}}$

* Makro $\max \int_0^T u(c)\, e^{-rt}\, dt$

K: kapital $\longrightarrow f(K) = C + K'$

## ② Vektorer og vektorregning

En n-vektor eller en kolonnevektor er definert som

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Vanlig notasjon for vektorer
$\underline{v}$ , boldface $\mathbf{v}$ , $\vec{v}$

### Regning med vektorer:

- addisjon (subtraksjon): $\underline{v} \pm \underline{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \pm \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 \pm w_1 \\ v_2 \pm w_2 \\ \vdots \\ v_n \pm w_n \end{pmatrix}$

- skalar multiplikasjon: $r \cdot \underline{v} = r \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ \vdots \\ rv_n \end{pmatrix} = \underline{v} \cdot r$

skalar = tall!

Eks: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+3 \\ 2+(-1) \end{pmatrix} = \underline{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}$ $\quad 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 \\ 3 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 2-(-1) \end{pmatrix} = \underline{\begin{pmatrix} -2 \\ 3 \end{pmatrix}}$

### Geometrisk tolkning:

Bruker $n=2$ for å gjøre det lettere å se geometrisk.

$P = (x_1, y_1)$
$Q = (x_2, y_2)$

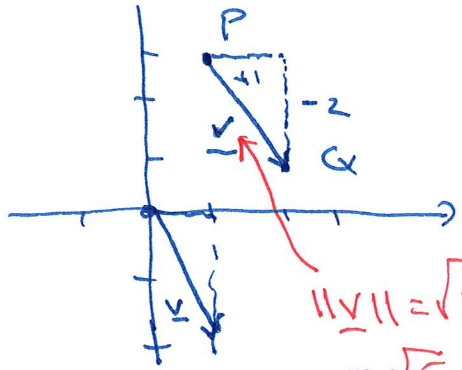
$$\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1) = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \underline{v}$$

vektor = forflytning

**Eks:** $P = (1,3)$

$Q = (2,1)$

$\vec{PQ} = \begin{pmatrix} 2-1 \\ 1-3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \underline{v}$




$\|\underline{v}\| = \sqrt{1^2 + (-2)^2} = \sqrt{5} \approx 2.23$

## Lengden til en vektor:

**Defn.** Lengden til vektoren

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{er} \quad \|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

## Lineærkombinasjoner av vektorer.

Anta at $\underline{v_1}, \underline{v_2}, \dots, \underline{v_r}$ har $n$ komponenter

**Defn** En lineærkombinasjon av $\underline{v_1}, \underline{v_2}, \dots, \underline{v_r}$ er et uttrykk på formen

$$c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} + \dots + c_r \cdot \underline{v_r}$$

der $c_1, c_2, \dots, c_r$ er vilkårlige skalarer.

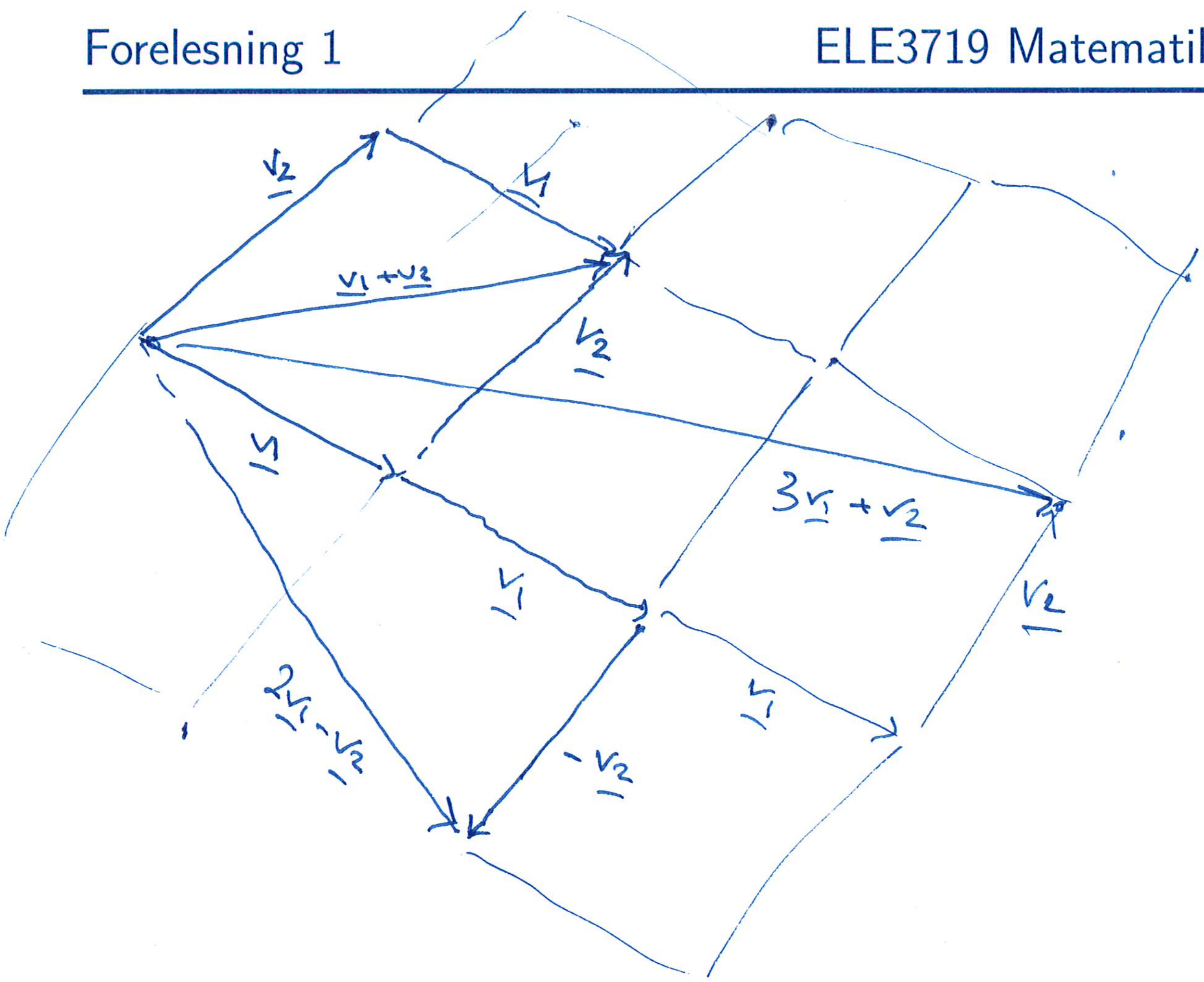
**Eks:**

$\underline{v_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ :

$2\underline{v_1} - \underline{v_2} = 2 \cdot \underline{v_1} + (-1) \cdot \underline{v_2}$

$= 2 \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix}}}$

$1 \cdot \underline{v_1} + 3\underline{v_2} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix}}}$




Merk: $-\underline{v}$ vektoren med same lengde som $\underline{v}$ og motsatt retning

$$\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{nullvektoren}$$

Defn: $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ er n-vektorer

$\text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r)$ = alle lineærkombinasjoner av vektorene $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$

$= \{ c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_r\underline{v}_r : c_1, c_2, \ldots, c_r$ er vilkårlige tall $\}$

Eks: $\underline{v_1} = \begin{pmatrix}1\\3\\-2\end{pmatrix}$ i) Hva er span$(\underline{v_1}, \underline{v_2})$ ?

$\underline{v_2} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$ ii) Ligger $\begin{pmatrix}1\\1\\2\end{pmatrix}$ i span$(\underline{v_1}, \underline{v_2})$ ?

$$\underline{c_1\underline{v_1} + c_2\underline{v_2} = \begin{pmatrix}1\\1\\2\end{pmatrix}} : \quad c_1\begin{pmatrix}1\\3\\-2\end{pmatrix} + c_2 \cdot \begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\begin{pmatrix}c_1 - c_2\\3c_1 + c_2\\-2c_1 + c_2\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

I $\quad c_1 - c_2 = 1$

II $\quad 3c_1 + c_2 = 1$

III $\quad -2c_1 + c_2 = 2$

I+II: $4c_1 = 2$

$c_1 = 2/4 = 1/2$

I : $1/2 - c_2 = 1$

$c_2 = 1/2 - 1 = -1/2$

$c_1 = 1/2, \; c_2 = -1/2$

III $\quad -2 \cdot \frac{1}{2} + (-1/2) = -3/2 \neq 2$

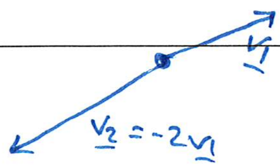
Ingen løsn. $\Rightarrow$ $\begin{pmatrix}1\\1\\2\end{pmatrix}$ er ikke i span$(\underline{v_1}, \underline{v_2})$.

<u>Defn.</u> Vi sier at $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ er <u>lineært avhengige</u> hvis minst én av vektorene er en lineærkombinasjon av de andre, og <u>lineært uavhengige</u> ellers.

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ n-vektor

<u>Eks:</u> $\underline{v_1} = \begin{pmatrix}1\\2\end{pmatrix}$ <u>Lineært avhengige</u> : $\underline{v_1} = c\underline{v_2}$ eller $\underline{v_2} = d\underline{v_1}$

$\underline{v_2} = \begin{pmatrix}-2\\-4\end{pmatrix}$ $\quad \underline{v_2} = -2\underline{v_1}$ eller $\underline{v_1} = -\frac{1}{2}\underline{v_2}$



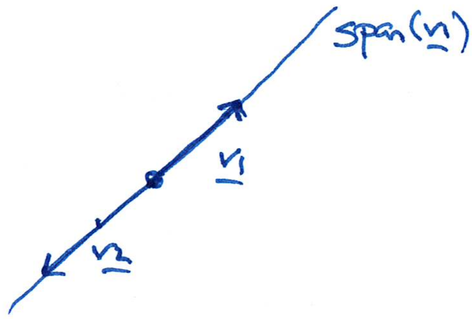
**Merk:** To vektorer $\underline{v_1}$ og $\underline{v_2}$ er lineært avhengige hvis og bare hvis de ligger langs samme rette linje.

Eks: $\underline{v_2} = -2\underline{v_1}$

Span$(\underline{v_1}, \underline{v_2})$:

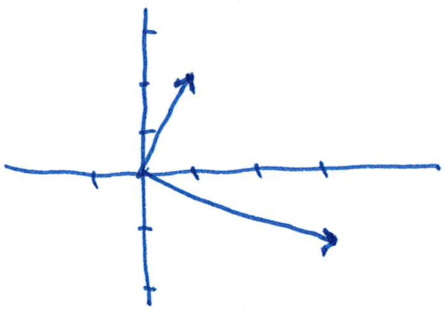
$$c_1 \cdot \underline{v_1} + c_2 \underline{v_2} = c_1 \underline{v_1} + c_2 \cdot (-2\underline{v_1})$$
$$= c_1 \underline{v_1} - 2c_2 \underline{v_1} = (c_1 - 2c_2) \cdot \underline{v_1} \leftarrow \text{span}(\underline{v_1})$$

$$\text{Span}(\underline{v_1}, \underline{v_2}) = \text{span}(\underline{v_1})$$




Eks: $\underline{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$   Ser at vektorene er lineært uavhengige

$\underline{v_2} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$



Hva er span$(\underline{v_1}, \underline{v_2})$?

$$c_1 \underline{v_1} + c_2 \underline{v_2} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \leftarrow \text{Ligger } \begin{pmatrix} 3 \\ 3 \end{pmatrix} \text{ i span}(\underline{v_1}, \underline{v_2})\text{?}$$

$$c_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\text{I} \quad c_1 + 3c_2 = 3$$
$$\text{II} \quad 2c_1 - c_2 = 3$$

I + 3II: $7c_1 = 12$

$c_1 = 12/7$

II $2 \cdot (12/7) - c_2 = 3$

$c_2 = \frac{24}{7} - \frac{21}{7} = \frac{3}{7}$

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \frac{12}{7} \cdot \underline{v_1} + \frac{3}{7} \cdot \underline{v_2}$$

Ja

Hva med $\begin{pmatrix} a \\ b \end{pmatrix}$: Ja

$$c_1 + 3c_2 = a$$
$$2c_1 - c_2 = b$$

$7c_1 = a + 3b$   $c_1 = \frac{a+3b}{7}$

$c_2 = \frac{2(a+3b)}{7} - \frac{21}{7} = \frac{2a+6b-21}{7}$

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Altså: $\text{span}(\underline{v}_1, \underline{v}_2)$ = alle vektorene i planet
$= \mathbb{R}^2$

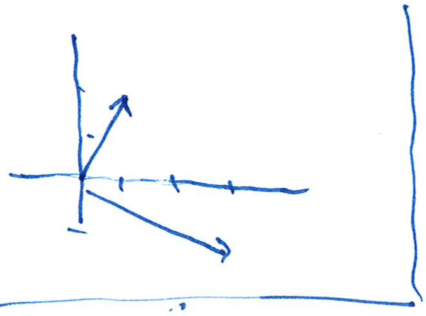
## 3 Indreprodukt og lengden til vektorer

Defn: $\underline{v}, \underline{w}$ n-vektorer

$$\underline{v} \cdot \underline{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = v_1 \cdot w_1 + v_2 w_2 + \dots + v_n \cdot w_n$$

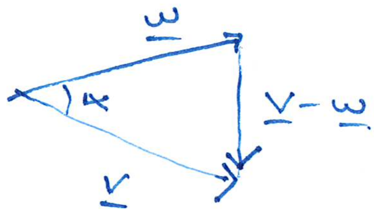
"indreprodukt, prikkprodukt, skalarprodukt", gir et tall som svar

Eks: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 1 \cdot 3 + 2 \cdot (-1) = 3 - 2 = \underline{1}$



Defn: Vektoren $\underline{v}$ og $\underline{w}$ kalles ortogonale hvis $\underline{v} \cdot \underline{w} = 0$. Det skrives i så fall $\underline{v} \perp \underline{w}$.

$\underline{v} \perp \underline{w}$ betyr at vinkelen er $90°$.




Egenskaper / regneregler:

i) $\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v}$

ii) $(\underline{v}_1 + \underline{v}_2) \cdot \underline{w} = \underline{v}_1 \cdot \underline{w} + \underline{v}_2 \cdot \underline{w}$

iii) $(r\underline{v}) \cdot \underline{w} = r \cdot (\underline{v} \cdot \underline{w})$

iv) $\underline{v} \cdot \underline{v} = \|\underline{v}\|^2$

$(\|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}})$

Vinkelen $\alpha = 90° \Leftrightarrow$ Pytagoras' holder

$$\|\underline{v}\|^2 + \|\underline{w}\|^2 = \|\underline{v} - \underline{w}\|^2$$

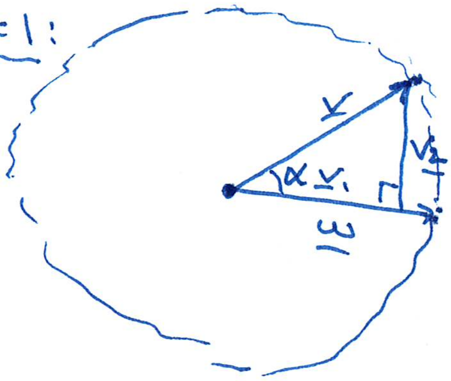
$$\|\underline{v} - \underline{w}\|^2 = (\underline{v} - \underline{w}) \cdot (\underline{v} - \underline{w})$$
$$= \underline{v} \cdot \underline{v} - \underline{w} \cdot \underline{v} + \underline{v} \cdot (-\underline{w}) + \underline{w} \cdot \underline{w}$$
$$= \|\underline{v}\|^2 + \|\underline{w}\|^2 - \underline{v} \cdot \underline{w} - \underline{v} \cdot \underline{w}$$
$$= \|\underline{v}\|^2 + \|\underline{w}\|^2 - 2(\underline{v} \cdot \underline{w}) = \|\underline{v}\|^2 + \|\underline{w}\|^2$$

$\underline{v} \cdot \underline{w} = 0$

$\Updownarrow$

Tolkning av $\underline{v} \cdot \underline{w}$ generelt:

Anta $\|\underline{w}\| = \|\underline{v}\| = 1$:




sirkel m/ radius = 1

$\underline{v} = \underline{v}_1 + \underline{v}_2$

slik at

$\underline{v}_1 = a\underline{w}$

$\underline{v}_2 \perp \underline{w}$

$\underline{v}_2 = \underline{v} - \underline{v}_1$

$= \underline{v} - a\underline{w}$

$\underline{v}_2 \perp \underline{w}$: $\underline{v}_2 \cdot \underline{w} = 0$

$(\underline{v} - a\underline{w}) \cdot \underline{w} = 0$

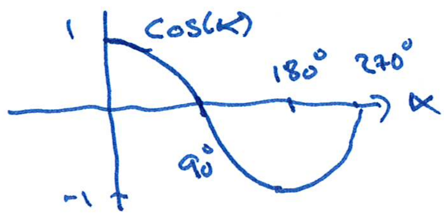
$\underline{v} \cdot \underline{w} - a\,\underline{w} \cdot \underline{w} = 0$

$\underline{v} \cdot \underline{w} - a \cdot 1 = 0$ $\quad a = \underline{v} \cdot \underline{w}$

Tolkning: ① Hvis $\|\underline{v}\| = \|\underline{w}\| = 1$ så er $-1 \leq \underline{v} \cdot \underline{w} \leq 1$

② Generelt er $-1 \leq \dfrac{\underline{v} \cdot \underline{w}}{\|\underline{v}\| \cdot \|\underline{w}\|} \leq 1$

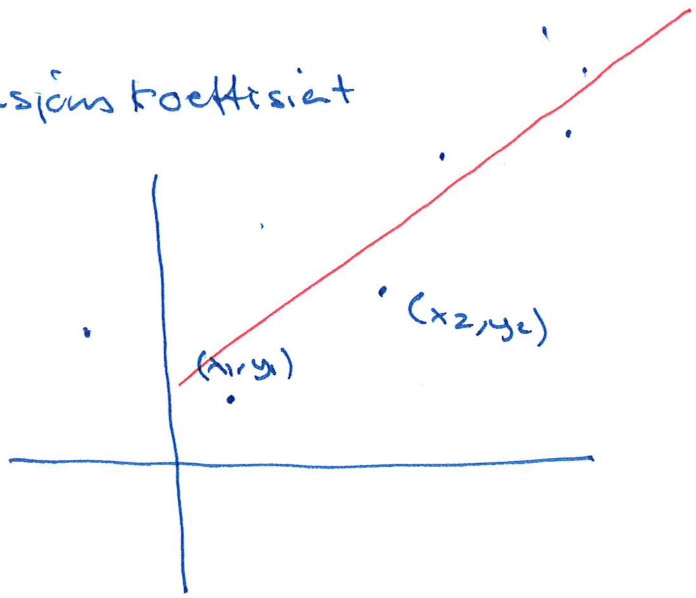
$$\frac{\underline{v} \cdot \underline{w}}{\|\underline{v}\| \cdot \|\underline{w}\|} = \cos(\alpha)$$



**Anvendelse:** Korrelasjonskoeffisient

| $x$ | $y$ |
|---|---|
| $x_1$ | $y_1$ |
| $x_2$ | $y_2$ |
| $\vdots$ | |
| $x_N$ | $y_N$ |

data m/ to variabler og $N$ observasjoner




$$\bar{x} = \frac{x_1 + x_2 + \dots + x_N}{N}$$

$$\bar{y} = \frac{y_1 + y_2 + \dots + y_N}{N}$$

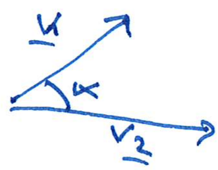
$$\underline{x} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_N - \bar{x} \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_N - \bar{y} \end{pmatrix}$$

$$\underline{v}_1 = \frac{1}{\|\underline{x}\|} \cdot \underline{x} \qquad \underline{v}_2 = \frac{1}{\|\underline{y}\|} \underline{y}$$

$$r = \underline{v}_1 \cdot \underline{v}_2 = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \cdot \|\underline{y}\|}$$ ← korrelasjons-koeff.

$$-1 \le r \le 1$$

$$r = \cos(\alpha)$$




**Eks:** $\underline{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\|\underline{v}\| = \sqrt{3^2 + 4^2} = 5$$

$$\frac{1}{\|\underline{v}\|} \cdot \underline{v} = \frac{1}{5} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$(3/5)^2 + (4/5)^2 = \frac{3^2 + 4^2}{5^2} = 1$$