Course paper 1 - EBA2911[1] - Mathematics for Business Analytics
15 Oct. – 22 Oct. 2021

SOLUTIONS

Problem 1

a) i) The sum is a geometric series which we read backwards. Then the first term is $a_1 = 6\,000 \cdot 1.0025^{25}$, the multiplication factor is $k = 1.0025$ and the number of terms is $n = 96 - 24 = 72$. The formula for the sum of a geometric series gives

$$6\,000 \cdot 1.0025^{25} \cdot \frac{1.0025^{72} - 1}{0.0025} = \underline{\underline{503\,122.08}}$$

ii) The sum read from left to right gives the the account balance after 8 years when 6000 is deposited every month for 6 years, that is 72 deposits, with the first deposit today, 3% nominal interest and monthly compounding with period rate $\frac{3\%}{12} = 0.0025$

b) If you borrow 1 000 000 today, pay back 15 000 each month for 10 years with first payment 4 years from now, r is the nominal interest and there is continuous compounding, then the left hand side of the equation gives the present value of the cash flow. The solution of the equation gives the internal rate of return r of the cash flow.

Problem 2

a) i) The mortgage is the present value of the cash flow, that is

$$\frac{10\,000}{\left(1+\frac{r}{12}\right)^{60}} + \frac{10\,000}{\left(1+\frac{r}{12}\right)^{61}} + \cdots + \frac{10\,000}{\left(1+\frac{r}{12}\right)^{419}}$$

This is a geometric series which we read backwards. Then $a_1 = \frac{10\,000}{(1+\frac{r}{12})^{419}}$, $k = 1 + \frac{r}{12}$ and $n = 419 - 59 = 360$. The formula for the sum of a geometric series gives

$$\frac{10\,000}{\left(1+\frac{r}{12}\right)^{419}} \cdot \frac{\left(1+\frac{r}{12}\right)^{360} - 1}{\frac{r}{12}} = \underline{\underline{\frac{120\,000}{\left(1+\frac{r}{12}\right)^{419}} \cdot \frac{\left(1+\frac{r}{12}\right)^{360} - 1}{r}}}$$

ii) If $r = 3\%$ this gives

$$\frac{120\,000}{\left(1+\frac{3\%}{12}\right)^{419}} \cdot \frac{\left(1+\frac{3\%}{12}\right)^{360} - 1}{3\%} = \frac{120\,000}{1.0025^{419}} \cdot \frac{1.0025^{360} - 1}{0.03} = \underline{\underline{2\,046\,994.83}}$$

If $r = 6\%$ this gives

$$\frac{120\,000}{\left(1+\frac{6\%}{12}\right)^{419}} \cdot \frac{\left(1+\frac{6\%}{12}\right)^{360} - 1}{6\%} = \frac{120\,000}{1.005^{419}} \cdot \frac{1.005^{360} - 1}{0.06} = \underline{\underline{1\,242\,729.39}}$$

b) With n payments the present value is

$$\frac{10\,000}{1{,}005^{60}} + \frac{10\,000}{1{,}005^{61}} + \cdots + \frac{10\,000}{1{,}005^{n+59}}$$

If we read this geometric series from left to right we get the sum

$$\frac{10\,000}{1.005^{60}} \cdot \frac{\left(\frac{1}{1.005}\right)^{n} - 1}{\left(\frac{1}{1.005}\right) - 1}$$

[1]Exam code EBA29101

If n is increasing without bounds, $\left(\frac{1}{1.005}\right)^n = \frac{1}{1.005^n}$ will become closer and closer to 0. Hence the present value of the regular cash flow without end date can be interpreted as

$$\frac{10\,000}{1.005^{60}} \cdot \frac{-1}{\left(\frac{1}{1.005}\right)-1} = \frac{10\,000}{1.005^{60}} \cdot \frac{1}{1-\left(\frac{1}{1.005}\right)} = \frac{10\,000}{1.005^{60}} \cdot \frac{1.005}{0.005} = \underline{\underline{1\,490\,158.11}}$$

which then is how much Kåre can borrow.

Problem 3

a) i) The present value of the cash flow is

$$-120 - \frac{170}{1.14^2} + \frac{100}{1.14^5} + \frac{200}{1.14^7} + \frac{250}{1.14^8} = \underline{\underline{-31.31}}$$

ii) The future value of the cash flow after 6 years is the present value multiplied with the growth factor for 6 years because

$$-31.31 \cdot 1.14^6 = \left(-120 - \frac{170}{1.14^2} + \frac{100}{1.14^5} + \frac{200}{1.14^7} + \frac{250}{1.14^8}\right) \cdot 1.14^6$$
$$= -120 \cdot 1.14^6 - 170 \cdot 1.14^4 + 100 \cdot 1.14 + \frac{200}{1.14} + \frac{250}{1.14^2}$$

which is the expression for the future value 6 years from now. This gives $-31{,}31 \cdot 1{,}14^6 = \underline{\underline{-68{,}72}}$ (you get $-68{,}71$ if you calculate the long sum).

b) With an extra payment of $\underline{\underline{68{,}72}}$ after 6 years the future value of the new cash flow becomes 0. But then the present value of the new cash flow is $\frac{0}{1{,}14^6} = 0$ by the same argument as for (a ii) and hence the IRR is 14%.

c) *Tall Cranes*' change is -50 eight years from now. That corresponds to $\frac{-50}{1{,}14^6} = \underline{\underline{-22{,}78}}$ two years from now. The payment therefore has to change from 170 to 147,22 two years from now for the IRR of this latest cash flow to be 14%.

Problem 4

a) Note that the presence of $\ln(x)$ makes the equation only defined for $x > 0$. A product of numbers equals 0 if and only if at least one of the factors is 0. This gives the alternatives

$$\begin{array}{ccc} e^x - 2 = 0 & \ln(x) - 3 = 0 & 4x^2 + 5x^3 = 0 \\ e^x = 2 & \ln(x) = 3 & x^2(4+5x) = 0 \\ x = \ln(2) & x = e^3 & x = 0 \quad \text{or} \quad x = -\frac{4}{5} \end{array}$$

with the solutions $\underline{\underline{x = \ln(2) \text{ or } x = e^3}}$ for the original equation.

b) We substitute $u = x^4$ into the equation and get $u^2 - 12u = 64$. By completing the square we get $(u-6)^2 = 64 + 36 = 100$ which gives $u = 6 \pm 10$, that is $u = -4$ or $u = 16$. By substituting back we get the equation $x^4 = -4$ which has no solutions and the equation $x^4 = 16$ with the solutions $\underline{\underline{x = \pm 2}}$.

c) We square each side and get $2x - 5 = (2-x)^2$, that is $2x - 5 = 4 - 4x + x^2$ which gives $x^2 - 6x = -9$. Completing the square gives $(x-3)^2 = -9 + 9 = 0$ with solution $\underline{x = 3}$. Because we started with an irrational equation we have to test this solution.
Inserted in the left h.s: $\sqrt{2 \cdot 3 - 5} = 1$. Inserted in the right h.s: $2 - 3 = -1$. Hence l.h.s. $\neq$ r.h.s.
<u>The equation has no solutions</u>.

Alternative argument: Since square roots are greater or equal to 0 we get $2 - x \geqslant 0$, that is $x \leqslant 2$. But then $2x - 5 \leqslant -1$ and $\sqrt{2x-5}$ is not defined.

Problem 5

a) We collect all terms into one fraction on the left hand side to get 0 on the right hand side:

$$\frac{4x+9}{x^2+2x+3}-2 \geqslant 0 \quad \text{expand the fraction } \frac{-2}{1}=\frac{-2(x^2+2x+3)}{x^2+2x+3}:$$

$$\frac{4x+9-2(x^2+2x+3)}{x^2+2x+3} \geqslant 0 \quad \text{resolve and collect terms in the numerator :}$$

$$\frac{-2x^2+3}{x^2+2x+3} \geqslant 0 \quad \text{and change sign: } \frac{2x^2-3}{x^2+2x+3} \leqslant 0$$

Because $x^2+2x+3=(x+1)^2+2 \geqslant 2$ the denominator is aways positive and thus the sign of the fraction is determined by the sign of the numerator $2x^2-3=2(x^2-1.5)=2(x-\sqrt{1.5})(x+\sqrt{1.5})$. We make a sign diagram and find that $-\sqrt{1.5} \leqslant x \leqslant \sqrt{1.5}$ which also can be written like this: $x \in \left[-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right]$

b) Note that we need $x>3$ for the inequality to be defined. By using the laws of logarithms the inequality can be written as

$$\ln\left(\frac{x(x-2)}{x-3}\right) \leqslant \ln 8$$

Because $\ln(x)$ is a strictly increasing function this inequality is equivalent to the inequality

$$\frac{x(x-2)}{x-3} \leqslant 8 \quad \text{Subtract 8 from both sides and expand the fraction:}$$

$$\frac{x(x-2)-8(x-3)}{x-3} \leqslant 0 \quad \text{Resolve and collect terms in the numerator:}$$

$$\frac{x^2-10x+24}{x-3} \leqslant 0 \quad \text{Find the roots } x=4 \text{ and } x=6 \text{ for the numerator and factorise:}$$

$$\frac{(x-4)(x-6)}{x-3} \leqslant 0$$

The original inequality is only defined if $x-3>0$ and hence the sign of the fraction equals the sign of the numerator. We make a sign diagram for $(x-4)(x-6) \leqslant 0$ and $4 \leqslant x \leqslant 6$ which also can be written as $x \in [4, 6]$

c) We note that both the numerator and the denominator separately are strictly increasing functions with one zero each: $\ln(e^{-2})+2=0$ and $e^{\ln(4)}-4=0$. We also note that the numerator is only defined for $x>0$. Then we can make a sign diagram:

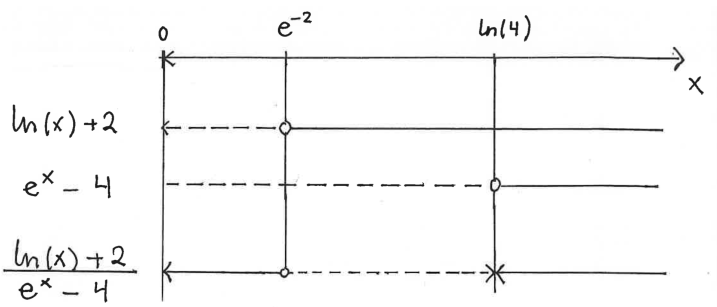

Figure 1: Sign diagram

We get $0<x \leqslant e^{-2}$ or $x>\ln(4)$ which also can be written as $x \in \langle 0, e^{-2}]$ or $x \in \langle \ln(4), \rightarrow \rangle$

Problem 6

a)

$$\begin{array}{l}
\big(\quad x^4+2x^3-28x^2+46x-21\big):\big(x-1\big)=x^3+3x^2-25x+21\\
\underline{-x^4\quad +x^3}\\
\qquad 3x^3-28x^2\\
\qquad \underline{-3x^3\quad +3x^2}\\
\qquad\qquad -25x^2+46x\\
\qquad\qquad \underline{25x^2-25x}\\
\qquad\qquad\qquad 21x-21\\
\qquad\qquad\qquad \underline{-21x+21}\\
\qquad\qquad\qquad\qquad 0
\end{array}$$

b) We have $f(x)=q(x)\cdot(x-t)+r$ where r has x-degree 0 (r consists only of numbers and t-s). We substitute $x=t$ into this equation and get $f(t)=q(t)\cdot 0+r=r$. Hence $\underline{\underline{r=t^4+2t^3-28t^2+46t-21}}$.

c) That the polynomial division $f(x):g(x)$ has 0 as remainder for a number t means by (b) that $f(t)=0$, i.e. we are looking for the roots of $f(x)$. From (a) we already have $\underline{x=1}$. Then we guess a root of $f(x):(x-1)=x^3+3x^2-25x+21$. For example $\underline{x=3}$ which gives $3^3+3\cdot3^2-25\cdot3+21=27+27-75+21=0$. Then we do the polynomial division

$$\begin{array}{l}
\big(\quad x^3+3x^2-25x+21\big):\big(x-3\big)=x^2+6x-7\\
\underline{-x^3+3x^2}\\
\qquad 6x^2-25x\\
\qquad \underline{-6x^2+18x}\\
\qquad\qquad -7x+21\\
\qquad\qquad \underline{7x-21}\\
\qquad\qquad\qquad 0
\end{array}$$

Finally we factorise $x^2+6x-7=\underline{(x-1)(x+7)}$. This gives $f(x)=(x-1)^2(x-3)(x+7)$. Hence t is either $\underline{\underline{1, 3 \text{ or } -7}}$ (and 1 is a double root).

Problem 7

In the standard form $f(x)=a(x-s)^2+d$ the d is the maximal value (if a is negative) and we read it off as $d=500$. From the graph we see the symmetry axis $x=s=200$ and hence $f(x)=a(x-200)^2+500$. From the graph we also see that $f(210)=495$. This gives the equation $a(210-200)^2+500=495$, that is $100a=-5$, that is $a=\frac{-5}{100}=-0.05$. Hence $\underline{f(x)=-0.05(x-200)^2+500}$. To find the roots we solve the equation $f(x)=0$, that is now $-0.05(x-200)^2+500=0$, i.e. $(x-200)^2=\frac{-500}{-0.05}=10\,000$ which gives $x=200\pm100$, that is $\underline{\underline{x=100 \text{ or } x=300}}$.

Problem 8

a) In the standard form $f(x)=a(x-s)^2+d$ the d is the maximal value (if a is negative) and it is given as $d=100$. Because P and Q has the same y-coordinate the symmetry axis $x=s$ has to be in the middle of the x-values, i.e. $s=10$. Hence $f(x)=a(x-10)^2+100$. Because Q is on the graph of $f(x)$, $f(12)=90$, that is $a(12-10)^2+100=90$ and we get $a=-2{,}5$. This gives $\underline{\underline{f(x)=-2{,}5(x-10)^2+100}}$.

b) The same argumentation as for $f(x)$ gives $g(x)=a(x-10)^2+d$. From $g(12)=90$ we get $a(12-10)^2+d=90$ which gives $a=\frac{90-d}{4}$ and hence $g(x)=\frac{90-d}{4}(x-10)^2+d$.

c) $g(x)$ has a maximum for $x = 10$ with maximal value d if $\frac{90-d}{4} \leqslant 0$, that is $\underline{\underline{d \geqslant 90}}$.

Problem 9

The standard form for an ellipse equation (with a and b positive numbers) is

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

Here the point (x_0, y_0) is the centre of the ellipse, a is the horizontal semi-axis, b is the vertical semi-axis. From the graph we find $(x_0, y_0) = (14, 12)$. Horizontal semi-axis is the horizontal distance from the centre to the ellipse, that is $a = 14-4 = 10$. Vertical semi-axis is the vertical distance from the centre to the ellipse, that is $b = 12-4 = 8$. Hence the equation of the ellipse is

$$\underline{\underline{\frac{(x-14)^2}{100} + \frac{(y-12)^2}{64} = 1}}$$

Problem 10

a) The standard form for a hyperbola function is $f(x) = c + \frac{a}{x-b}$. Vertical asymptote is $x = b$ and horizontal asymptote is $y = c$. The point $S = (b, c)$ is the symmetry point of the hyperbola. It seems to be $(20, 15)$. If we draw the line l through the points $(17, 14)$ and $(23, 16)$ on the graph and the line l' through the points $(19, 12)$ and $(21, 18)$ on the graph we see that the intersection point of l with l' is S and the four points have the same distance from S. Hence $S = (20, 15)$. Then $f(x) = 15 + \frac{a}{x-20}$. To determine a we can use the equation $f(21) = 18$, that is $15 + \frac{a}{21-20} = 18$ which gives $a = 3$ and then $\underline{\underline{f(x) = 15 + \frac{3}{x-20}}}$.

b) To find the intersection points we first determine the x-values which give the same y-values for the two functions, i.e. we solve the equation $f(x) = g(x)$:

$$\begin{aligned}
15 + \frac{3}{x-20} &= 16 - \frac{3}{x-12} && \text{move everything to one side:}\\
15 + \frac{3}{x-20} - 16 + \frac{3}{x-12} &= 0 && \text{and expand the fractions:}\\
\frac{-(x-20)(x-12) + 3(x-12) + 3(x-20)}{(x-20)(x-12)} &= 0 && \text{resolve:}\\
\frac{-x^2 + 32x - 240 + 3x - 36 + 3x - 60}{(x-20)(x-12)} &= 0 && \text{collect terms:}\\
\frac{-x^2 + 38x - 336}{(x-20)(x-12)} &= 0
\end{aligned}$$

For a fraction to be 0 the numerator has to be 0 and the denominator not 0. We therefore solve $-x^2 + 38x - 336 = 0$, that is $x^2 - 38x = -336$. Completing the square: $(x-19)^2 = 19^2 - 336 = 25$ gives $x = 19 \pm 5$, that is $\underline{x = 14}$ or $\underline{x = 24}$ (which are not zeros in the denominator). Then we calculate the y-values: $f(14) = 15 + \frac{3}{14-20} = 14.5$ (which equals $g(14)$) and $f(24) = 15 + \frac{3}{24-20} = 15.75$ (which equals $g(24)$). The intersection points between the hyperbolas are then $\underline{\underline{(14, 14.5)}}$ and $\underline{\underline{(24, 15.75)}}$. See figure 2.

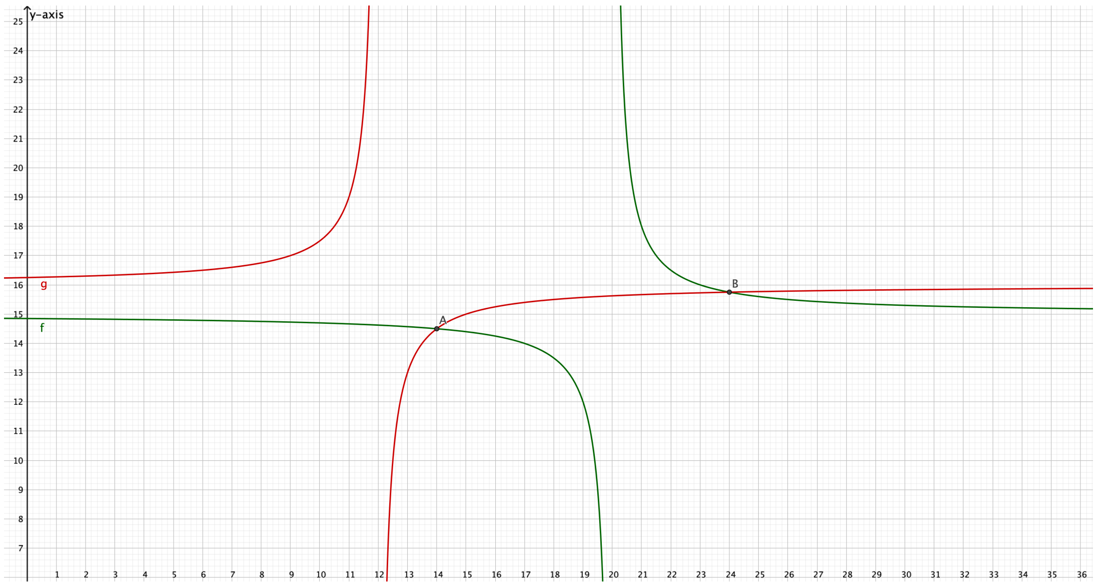

Figure 2: Hyperbolas

Problem 11

a) We put $y = \sqrt{x-1} + 3$ and solve for x. First we subtract 3 from both sides (b.s.): $y - 3 = \sqrt{x-1}$ and square b.s: $(y-3)^2 = x - 1$. Add 1 on b.s: $x = (y-3)^2 + 1$ and change the variables which gives $\underline{\underline{g(x) = (x-3)^2 + 1}}$. We have $V_g = D_f = \underline{\underline{[1, 26]}}$ and $D_g = V_f$. Because $f(x)$ is a increasing function the minimum value is $f(1) = \sqrt{1-1} + 3 = 3$ while the maximum value is $f(26) = \sqrt{26-1} + 3 = 8$. By the intermediate value theorem all values between 3 and 8 will be in the range of $f(x)$ so $\underline{\underline{D_g = [3, 8]}}$.

b) We put $y = e^{-0,1x+2} + 5$ and solve for x. First we subtract 5 on b.s: $y - 5 = e^{-0,1x+2}$. Insert b.s. in $\ln(-)$ and get $\ln(y-5) = -0,1x + 2$. Subtract 2 from b.s: $\ln(y-5) - 2 = -0,1x$. Multiply with -10 on b.s: $20 - 10\ln(y-5) = x$ and change the variables which gives $\underline{\underline{g(x) = 20 - 10\ln(x-5)}}$. We have $V_g = D_f = \underline{\underline{[10, \rightarrow\rangle}}$ and $D_g = V_f$.
Because $f(x)$ is a decrasing function the maximum value is $f(10) = e^{-0,1\cdot 10+2} + 5 = e + 5$ while $f(x)$ decreases towards 5 when x increases without bounds (becuase $e^{-0,1x+2} = \frac{e^2}{e^{0,1x}}$ and the denominator increases without bounds), without being 5 for any x. By the intermediate value theorem all values between $e + 5$ and 5 apart from 5 itself will be in the range of $f(x)$ and so $\underline{\underline{D_g = \langle 5, e+5]}}$.