EVALUATION GUIDELINES - Course paper

EBA 29101
Mathematics for Business Analytics

Department of Economics

Start date:	09.10.2020	Time 09:00
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For more information about formalities, see examination paper.

Term paper - EBA2911[1] Mathematics for Business Analytics

9 Oct. – 16 Oct. 2020

SOLUTIONS

Problem 1

a) We read the geometric series from right to left. Then the first term is $a_1 = \frac{10\,000}{1.01^{215}}$, the multiplication factor is $k = 1.01$ and the number of terms is $n = 215 - 35 = 180$. The formula for the sum of a geometric series gives

$$a_1 \cdot \frac{k^n - 1}{k - 1} = \frac{10\,000}{1.01^{215}} \cdot \frac{1.01^{180} - 1}{0.01} = \underline{\underline{588\,179{,}46}}$$

This sum can represent the present value of a cash flow of 10 000 every month for 15 years with the first payment 3 years form now and 12% nominal interest with monthly compounding.

b) If the situation is as described in (a), but with continuous compuonding the monthly discount factor is $e^{0.01}$. Then 10 000 payed 36 months from now has present value

$$\frac{10\,000}{(e^{0.01})^{36}} = \frac{10\,000}{e^{0.36}}$$

and so on. We obtain the given sum as the present value of the same cash flow with the same nominal interest, but with continuous compounding.

Problem 2

(a) The present value is

$$-30 - \frac{30}{1.15^2} + \frac{40}{1.15^8} + \frac{40}{1.15^9} + \frac{40}{1.15^{10}} = \underline{\underline{-18.35}}$$

(b) The future value after 7 years is

$$-30 \cdot 1.15^7 - 30 \cdot 1.15^5 + \frac{40}{1.15} + \frac{40}{1.15^2} + \frac{40}{1.15^3} = \underline{\underline{-48.81}}$$

Note that this is the same as the present value multiplied with the growth factor for 7 years: $-18.35 \cdot 1.15^7 = -48.81$.

(c) The future value of the cash flow after 7 years is -48.81, hence if Hege adds a payment of $\underline{\underline{48.81}}$ 7 years from now, the future value of the cash flow 7 years from now will be 0. This implies that the internal rent of return is 15% because the present value of the new cash flow equals $\frac{0}{1.15^7} = 0$.

Problem 3

a) If a product of numbers equals zero at least one of the factors equals zero. We get

$$\begin{aligned} x - 2 &= 0 &\quad\text{or}\quad 4x - 7 &= 0 &\quad\text{or}\quad 9 + x &= 0 \\ x &= \underline{\underline{2}} &\quad\text{or}\quad x &= \underline{\underline{\frac{7}{4}}} &\quad\text{or}\quad x &= \underline{\underline{-9}} \end{aligned}$$

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b) We substitute $u = x^3$ and get a quadratic equation $u^2 - 6u = 16$, complete the square and obtain the equation $(u-3)^2 = 16 + 3^2 = 25$. Hence $u = 3 - 5 = -2$ or $u = 3 + 5 = 8$. We substitute back and get $x^3 = -2$ or $x^3 = 8$ which give $x = \sqrt[3]{-2} = \underline{\underline{-\sqrt[3]{2}}}$ or $x = \sqrt[3]{8} = \underline{\underline{2}}$.

c) We isolate one of the roots by adding $\sqrt{x-7}$ on each side:

$$\sqrt{3x+4} = \sqrt{x-7} + 5$$

Square each side:

$$3x + 4 = x - 7 + 10\sqrt{x-7} + 25$$

Collect terms and isolate the other root $10\sqrt{x-7}$:

$$2x - 14 = 10\sqrt{x-7}$$

Divide each side by 2:

$$x - 7 = 5\sqrt{x-7}$$

Then square each side

$$x^2 - 14x + 49 = 25x - 175$$

and simplify:

$$x^2 - 39x = -224$$

Complete the square:

$$\left(x - \frac{39}{2}\right)^2 = -224 + \frac{39^2}{2^2} = \frac{625}{4} = \frac{25^2}{2^2}$$

This gives two possible solutions

$$\underline{x = 7} \qquad \text{og} \qquad \underline{x = 32}\,.$$

Since we have squared each side of the equation (in fact twice) we have to check if the two possible solutions in fact are solutions of the original equation:

With $x = 7$ the left hand side is $\sqrt{3 \cdot 7 + 4} - \sqrt{7 - 7} = 5 - 0 = 5$ which equals the right hand side and so $\underline{\underline{x = 7}}$ is a solution.

With $x = 32$ the left hand side is $\sqrt{3 \cdot 32 + 4} - \sqrt{32 - 7} = 10 - 5 = 5$ which equals the right hand side and so $\underline{\underline{x = 32}}$ is a solution.

d) We substitute $u = e^{0.2x}$ and get the equation

$$\frac{u}{u - 10} = 11$$

Multiplying with $u - 10$ on each side (we assume $u \neq 10$):

$$u = 11(u - 10) = 11u - 110 \quad \text{som gir} \quad 10u = 110 \qquad \text{i.e.} \qquad \underline{u = 11}$$

Substitute back and get the equation $e^{0.2x} = 11$. Insert the left hand side and the right hand side into ln( ) and get

$$\ln(e^{0.2x}) = \ln(11) \qquad \text{i.e.} \qquad 0.2x = \ln(11) \qquad \text{i.e.} \qquad \underline{\underline{x = 5 \cdot \ln(11)}}$$

(and then $u = e^{\ln(11)} = 11 \neq 10$).

e) By the rules of logarithms we have $\ln(x) - \ln(x-3) = \ln\left(\frac{x}{x-3}\right)$. The equation is then

$$\ln\left(\frac{x}{x-3}\right) = 1.12$$

Insert the left hand side and the right hand side into $e^{(\,)}$ and obtain

$$e^{\ln(\frac{x}{x-3})} = e^{1.12} \qquad \text{i.e.} \qquad \frac{x}{x-3} = e^{1.12}$$

Multiplying with $(x-3)$ on each side (we assume $x \neq 3$)

$$x = e^{1.12} \cdot x - 3e^{1.12}$$

Collect the x-terms on the same side of the equation

$$(e^{1.12} - 1)x = 3e^{1.12}$$

Divide by $(e^{1.12} - 1)$ on each side and get

$$\underline{\underline{x = \frac{3e^{1.12}}{e^{1.12} - 1}}}$$

which is larger than 3.

Problem 4

a) The right hand side is 0 and hence we can use a sign diagram after we have factorised the numerator. We complete the square

$$x^2 - 4x + 5 = (x-2)^2 - 2^2 + 5 = (x-2)^2 + 1$$

and observe that the expression is greater or equal to 1. Hence no further factorisation is possible. But that doesn't matter. We now know that the numerator is positive for all x and hence the sign of the denominator determines the sign of the fraction. We get $\underline{\underline{x > 4}}$ (note that $x = 4$ is excluded since it makes the denominator of the fraction equal to 0).

b) Here we don't have 0 on the right hand side and then we cannot use the sign diagram as it is. Remove 1 on each side and multiply upstairs and downstairs with the denominator of the other equation:

$$\frac{2x-12}{(x-3)(x+4)} - 1 \cdot \frac{(x-3)(x+4)}{(x-3)(x+4)} \geqslant 0$$

We write it as one fraction since the denominators are equal. We also resolve the parentheses in numerator:

$$\frac{2x-12-(x^2+x-12)}{(x-3)(x+4)} \geqslant 0$$

We collect terms:

$$\frac{-x^2+x}{(x-3)(x+4)} \geqslant 0$$

Here x is a common factor in the numerator:

$$\frac{x(-x+1)}{(x-3)(x+4)} \geqslant 0$$

If we don't like the sign we can multiply the inequality with -1, but then we have to turn the inequality:

$$\frac{x(x-1)}{(x-3)(x+4)} \leqslant 0$$

Now we are ready to use a sign diagram.

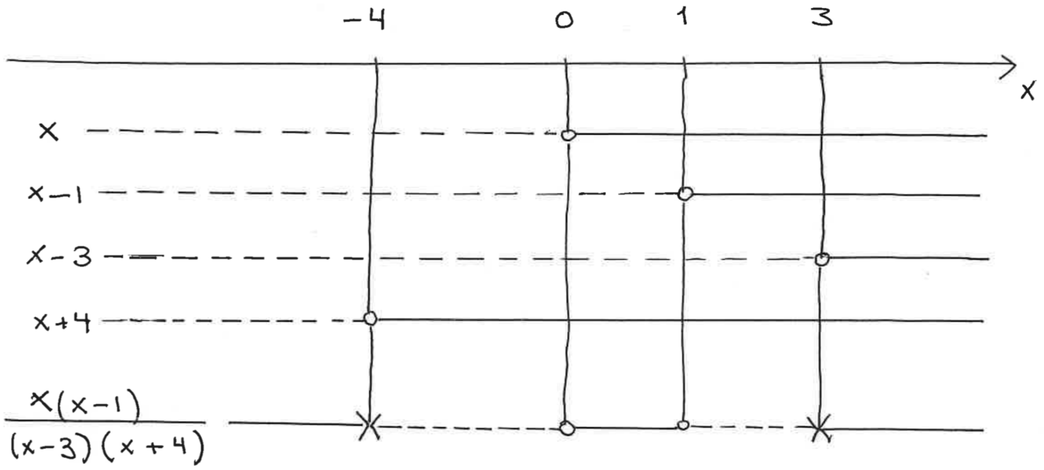

Figure 1: Sign diagram in 2b

That is

$$\underline{\underline{-4 < x \leqslant 0 \quad \text{or} \quad 1 \leqslant x < 3}}$$

Alternate way of writing: $\underline{\underline{x \in \langle -4, 0] \text{ or } x \in [1, 3\rangle}}$.

c) The exponent of e should not be greater than $\ln(20)$ so we get the inequalty $-0{,}1x \leqslant \ln(20)$. (Said differently: Because $\ln(x)$ is a strictly increasing function we can insert the left hand side and the right hand side into ln( ) and get an equivalent inequality.) If we multiply each side by -10 we get the inequality and solution $\underline{\underline{x \geqslant -10 \cdot \ln(20)}}$. Alternate way of writing: $\underline{\underline{x \in [-10 \cdot \ln(20), \infty\rangle}}$.

d) What is inside of ln should not be greater than e^3 (because e^x er is a strictly increasing function we can insert the left hand side and the right hand side into $e^{(\,)}$ and get an equivalent inequality) and at the same time greater than 0 (ln is only defined for positive numbers). This gives the two inequalities $0 < x - 1 \leqslant e^3$. We add 1 in all the three expressions and get the inequalities and the answer $\underline{\underline{1 < x \leqslant e^3 + 1}}$. Alternate way of writing: $\underline{\underline{x \in \langle 1\,, e^3 + 1]}}$.

Problem 5

a) We have to multiply the factors in the denominator to be able to do the polynomial division: $x(x-1)(x-5) = x^3 - 6x^2 + 5x$. Then

$$\begin{array}{l} (\quad x^4 - 14x^3 + 53x^2 - 40x - 1) : (x^3 - 6x^2 + 5x) = x - 8 + \dfrac{-1}{x^3 - 6x^2 + 5x} \\ \underline{-x^4 \;\; + 6x^3 \;\; - 5x^2} \\ \qquad -8x^3 + 48x^2 - 40x \\ \qquad \underline{\;\;\; 8x^3 - 48x^2 + 40x} \end{array}$$

Hence the remainder is $\underline{\underline{-1}}$.

b) We read off the vertical asymptotes as the x-values where the denominator in the remainder is zero (and the numerator is -1 which of course is different from 0 for all x). This gives the vertical lines $\underline{\underline{x = 0,\ x = 1 \text{ og } x = 5}}$ (y is free). Moreover $\underline{\underline{y = x - 8}}$ is a non-vertical (oblique) asymptote since

$$\frac{-1}{x^3 - 6x^2 + 5x} \xrightarrow[x \to \pm\infty]{} 0$$

Problem 6

All quadratic polynomials can be written as $f(x) = a(x-s)^2 + d$ for some numbers a, s and d. The plan is to find this expression. Then we solv the equation $f(x) = 0$. We know that the vertical line $x = s$ is the asymptote for $f(x)$. We read off the graph that $s = 17$. Since the parabola i «sad», a is negative and d hence is the maximum value of $f(x)$ (for $x = 17$). We read off the graph that $d = 110$. From the graph it also seems that $f(7) = 105$, that is $a(7-17)^2 + 110 = 105$, and then $100a = -5$, so $a = -0.05$. Hence

$$f(x) = -0{,}05(x-17)^2 + 110$$

To find the roots (zeros) we solve the equation $-0.05(x-17)^2 + 110 = 0$, that is $(x-17)^2 = 2200$ which gives $\underline{\underline{x = 17 \pm 10\sqrt{22}}}$.

Problem 7

a) An ellipse has a standard equation $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$. Here $(x_0\,, y_0)$ is the centre of the ellipse. From the graph we see that $(x_0\,, y_0) = (5\,, 3)$. Moreover, a is the length of the horizontal semi-axis, that is the horizontal distance from the centre to the ellipse. We see that $a = 10 - 5 = 5$. Correspondingly, b is the length of the vertical semi-axis. We see that $b = 6 - 3 = 3$. Then the equation is

$$\underline{\underline{\frac{(x-5)^2}{25} + \frac{(y-3)^2}{9} = 1}}$$

b) The one-point formula gives the linear equation of the line L as $y-3=-0.3(x-10)$, i.e. $y=-0.3x+6$. We substitute y in the ellipse equation with this expression:

$$\frac{(x-5)^2}{25}+\frac{(-0.3x+6-3)^2}{9}=1$$

Multiplying both sides with $25\cdot 9=225$ gives

$$9(x^2-10x+25)+25(0.09x^2-1.8x+9)=225$$

Multiplying into the parantheses and collecting terms gives the quadratic equation

$$11.25x^2-135x=-225$$

Dividing each side by 11.25 gives

$$x^2-12x=-20$$

Completing the square:

$$(x-6)^2=-20+6^2=16$$

We get $x=6\pm 4$, so $x=2$ is the new root and then $y=-0.3\cdot 2+6=5.4$. Hence the other intersection point between the ellipse and the line L is $\underline{\underline{(2\,,5.4)}}$.

Problem 8

All hyperbola functions can be written as $f(x)=c+\frac{a}{x-b}$ for some numbers a, b and c. Here the vertical line $x=b$ (and y free) is the vertical asymptote to $f(x)$. Looking at the graph it seems that $b=10$. Furthermore, the horizontal line $y=c$ (and x free) is the horizontal asymptote for $f(x)$. Looking at the graph it seems that $c=7$. To determine a we find a point on the graph, e.g. $(9\,,8)$. Then $7+\frac{a}{9-10}=8$ which gives $a=-1$. Hence

$$f(x)=\underline{\underline{7-\frac{1}{x-10}}}$$

Problem 9

a) We put $y=f(x)$ and solve for x. I.e. $y=-0.5x+10$ which gives $x=-2y+20$. Hence the inverse function has the expression $g(x)=\underline{\underline{-2x+20}}$. The domain of definition D_g is as always equal to the range of $f(x)$. Because $f(x)$ is a decreasing function, the maximum value is $f(0)=10$ and the minimum value is $f(20)=0$. Hence $D_g=R_f=\underline{\underline{[0\,,10]}}$. Finally, the range of $g(x)$ is as always equal to the domain of $f(x)$, i.e. $R_g=\underline{\underline{[0\,,20]}}$.

b) We put $y=2\ln(x+3)-1$ and solve for x. We add 1 to each side and divide by 2 on each side. It gives

$$\ln(x+3)=\frac{y+1}{2}$$

We insert both sides into $e^{()}$ and obtain

$$e^{\ln(x+3)}=e^{\frac{y+1}{2}} \quad \text{i.e.} \quad x+3=e^{\frac{y+1}{2}} \quad \text{i.e.} \quad x=e^{0.5y+0.5}-3$$

Hence the inverse function has the expression

$$g(x)=\underline{\underline{e^{0.5x+0.5}-3}}$$

Since $f(x)$ is a strictly increasing function approaching $-\infty$ when x approaches 0 from above and $f(x)$ grows without bounds when x approaches $+\infty$, we have that $R_f=$ the whole number line and so $D_g=$ $\underline{\underline{\text{the whole number line}}}$. Moreover, $R_g=D_f=\underline{\underline{\langle -3\,,\infty\rangle}}$.

Problem 10

a) i) We put the IRR equal to r. The present value of the cash flow is the sum of the present values of the payments. Since the present value is supposed to be 0 we get the equation

$$-10+\frac{18}{e^{3r}}=0 \quad \text{which gives} \quad e^{3r}=\frac{18}{10}=\frac{9}{5}$$

Inserted into ln( ) this gives the equation $3r=\ln(9)-\ln(5)$ and $\underline{\underline{r=\frac{1}{3}[\ln(9)-\ln(5)]=19.593\%}}$.

ii) Here we get the equation

$$-\frac{10}{e^{5r}}+\frac{18}{e^{8r}}=0 \quad \text{which gives} \quad \frac{e^{8r}}{e^{5r}}=\frac{18}{10} \quad \text{i.e.} \quad e^{3r}=\frac{9}{5}$$

This is the same equation as in (i) so the answer is the same: $\underline{\underline{r=\frac{1}{3}[\ln(9)-\ln(5)]=19.593\%}}$.

iii) Here we get the equation

$$-\frac{10}{e^{5r}}+\frac{18}{e^{11r}}=0 \quad \text{which gives} \quad \frac{e^{11r}}{e^{5r}}=\frac{18}{10} \quad \text{i.e.} \quad e^{6r}=\frac{9}{5}$$

This gives $6r=\ln(9)-\ln(5)$ and $\underline{\underline{r=\frac{1}{6}[\ln(9)-\ln(5)]=9.796\%}}$.

b) Now we do the same deduction with undetermined parametres. Since the present value of the cash flow should be 0 we get the equation

$$-\frac{A}{e^{mr}}+\frac{B}{e^{nr}}=0 \quad \text{which gives} \quad \frac{e^{nr}}{e^{mr}}=\frac{B}{A} \quad \text{i.e.} \quad e^{(n-m)r}=\frac{B}{A}$$

Inserted in ln( ) this gives $(n-m)r=\ln(B)-\ln(A)$ and $\underline{\underline{r=\frac{1}{(n-m)}[\ln(B)-\ln(A)]}}$.

c) When A $(=10)$ and B $(=18)$ are fixed numbers we see that the IRR is given as a fraction with a fixed number in the numerator (namely $\ln(B)-\ln(A)=\ln(18)-\ln(10)$) and in the denominator $n-m$ which is the length of the interval between the two payments. We see that this happens in (ai-ii).

Twice as large a time interval means that we have to divide by a number twice as large. Then the fraction is half as large and so is the IRR. We see that this happens in (aiii).