EBA2911, lecture 9, 11 oct. 2021, Runar Ile

Plan

1. Rational functions and asymptotes
2. Hyperbolas
3. Continuity and the intermediate value theorem

1. Rational functions and asymptotes

Rational function $f(x) = \frac{p(x)}{q(x)}$ ← polynomials

Ex $f(x) = \frac{2x+1}{x^2+3}$ - would like to see what happens when x is big.

- divide by x^2 both in the numerator and in the denominator

$$= \frac{\frac{2x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}} = \frac{\frac{2}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}} \xrightarrow[x \to \pm\infty]{} \frac{0}{1} = 0$$

$$f(1000) = \frac{\frac{2}{1000} + \frac{1}{1000^2}}{1 + \frac{3}{1000^2}} = 0.00200099\ldots$$

This means that the line $y=0$ (x free) is a horizontal asymptote for $f(x)$.

Ex $f(x) = \frac{2x+1}{(x-1)(x-5)}$ $(x \neq 1,\ x \neq 5)$

If $x \to 1^-$ "x is approaching 1 from below"

$x = 0.9$, $x = 0.99$, $x = 0.999$

then

$$\left.\begin{aligned} x-1 &\to 0^- \\ x-5 &\to -4^- \\ 2x+1 &\to 3^- \end{aligned}\right\} \text{implies } f(x) = \frac{\overbrace{(2x+1)}^{\to 3^-}}{\underbrace{(x-1)}_{\to 0^-}\underbrace{(x-5)}_{\to -4^-}} \xrightarrow[x \to 1^-]{} +\infty$$

If $x \to 1^+$ then e.g. $x = 1.1$, $x=1.01$, $x=1.001$

$$\left.\begin{array}{l} x-1 \longrightarrow 0^+ \\ x-5 \longrightarrow -4^+ \\ 2x+1 \longrightarrow 3^+ \end{array}\right\} \text{implies } f(x) = \frac{\overbrace{(2x+1)}^{3^+}}{\underbrace{(x-1)}_{0^+}\underbrace{(x-5)}_{-4^+}} \xrightarrow[x \to 1^+]{} -\infty$$

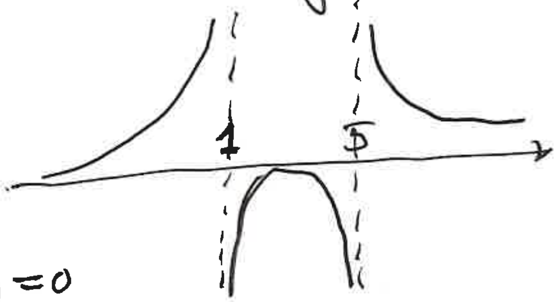
The line $x=1$ (y free) is a vertical asymptote for $f(x)$

Similarily: $f(x) \xrightarrow[x \to 5^+]{} +\infty$

$$\left\{\begin{array}{l} 2x+1 \longrightarrow 11^+ \\ x-1 \longrightarrow 4^+ \\ x-5 \longrightarrow 0^+ \end{array}\right.$$

$f(x) \xrightarrow[x \to 5^-]{} -\infty$

The line $x=5$ (y free) is a vertical asymptote for $f(x)$



<u>Note</u> $f(x)$ also has a horizontal asymptote $y=0$

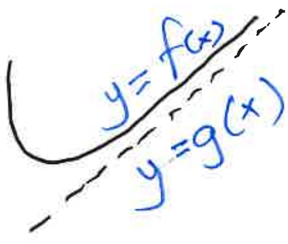
<u>Non-vertical asymptotes</u>

<u>Ex</u> $f(x) = x - 5 + \dfrac{2}{x-4}$ has a vertical asymptote $x=4$

Put $g(x) = x-5$.

Then the graph of $f(x)$ is approaching the graph of $f(x)$ when $x \longrightarrow \pm\infty$

Then $g(x)$ is a non-vertical asymptote for $f(x)$ because



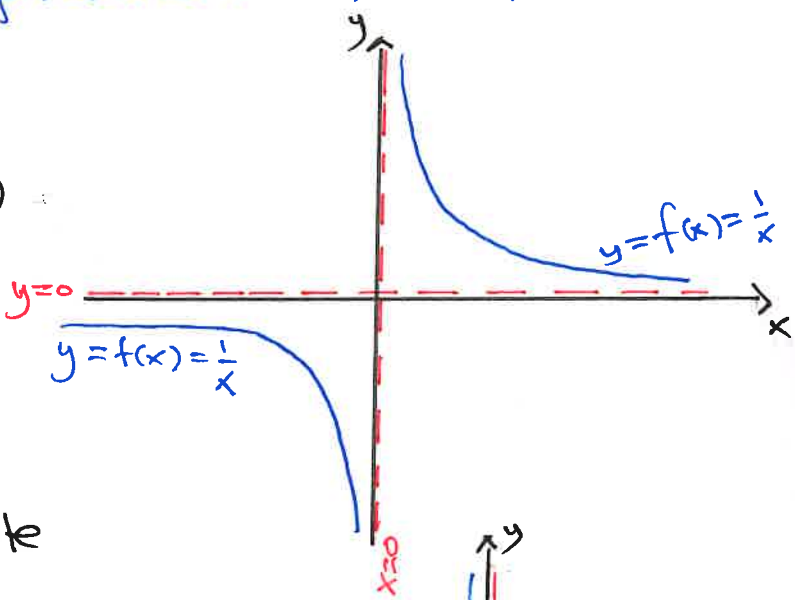
$$f(x) - g(x) = \frac{2}{x-4} \xrightarrow[x \to \pm\infty]{} 0$$

Note that $f(x) = \dfrac{(x-5)(x-4)+2}{(x-4)} = \dfrac{x^2-9x+22}{x-4}$

have to do polynomial division to find the better expression for $f(x)$.

2. Hyperbolas

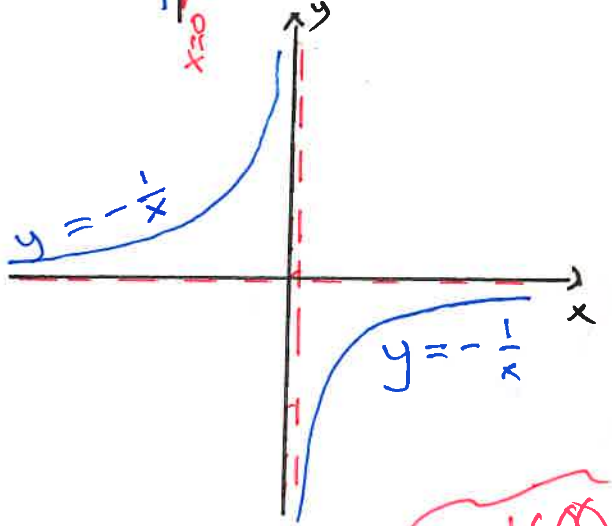
<u>Ex</u> $f(x) = \dfrac{1}{x}$ $(x \neq 0)$



The line $x=0$ is a vertical asymptote

The line $y=0$ is a horizontal asymptote

<u>Ex</u> $f(x) = -\dfrac{1}{x}$ $(x \neq 0)$



Has the same asymptotes.

<u>Definition</u> A function $f(x)$ is a hyperbola function if it can be written as

$$f(x) = c + \frac{a}{x-b} \quad (a \neq 0)$$

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<u>Ex</u> $f(x) = \dfrac{3x-5}{x-2}$ is a hyperbola function because

$$(3x-5) : (x-2) = 3 + \frac{1}{x-2}$$

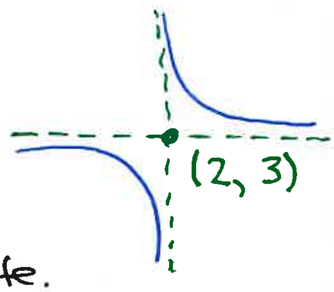
$$\begin{array}{l} -(3x-6) \\ \hline \quad 1 \end{array}$$

$\cdot(x-2)$

so $a = 1$, $b = 2$, $c = 3$

We have $3 + \dfrac{1}{x-2} \xrightarrow[x \to 2^-]{} -\infty$

$3 + \dfrac{1}{x-2} \xrightarrow[x \to 2^+]{} +\infty$



So the line $x = 2$ is a vertical asymptote.

Also note that $3 + \dfrac{1}{x-2} \xrightarrow[x \to \pm\infty]{} 3$

So the line $y = 3$ is a horizontal asymptote.

$f(1) = 3 + \dfrac{1}{1-2} = 2$

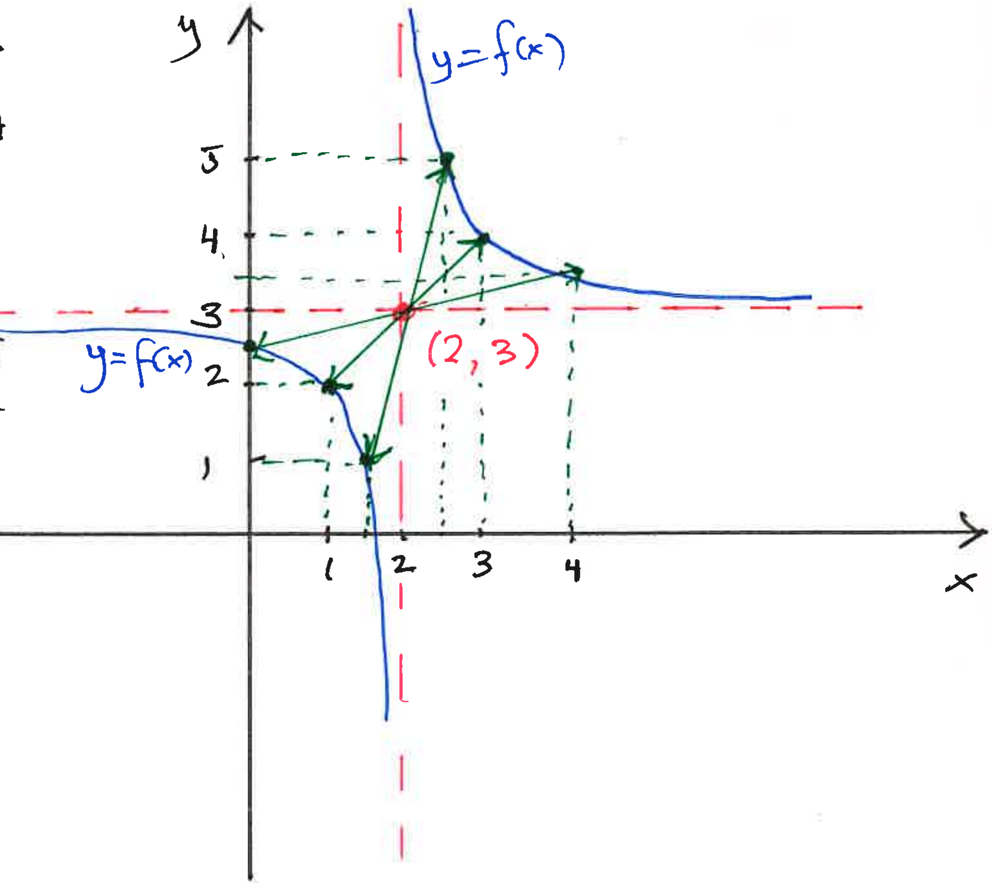
$f(3) = 3 + \dfrac{1}{3-2} = 4$

$f(1.5) = 3 + \dfrac{1}{1.5-2} = 1$

$f(2.5) = 3 + \dfrac{1}{2.5-2} = 5$

$f(0) = 3 + \dfrac{1}{0-2} = 2.5$

$f(4) = 3 + \dfrac{1}{4-2} = 3.5$



The graph of a hyperbola function is symmetric through the intersection point of the asymptotes.

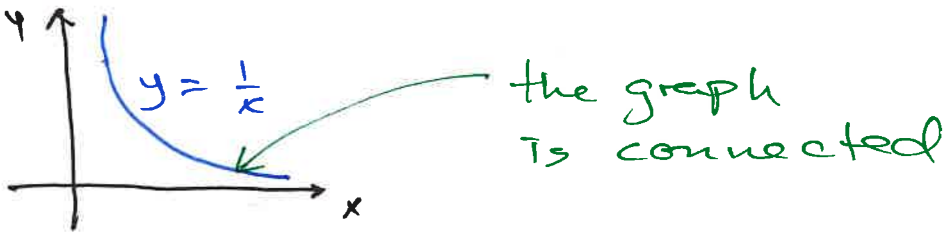
3. Continuity and the intermediate value theorem

A function is continuous if the graph is connected for every interval in the domain of definition.

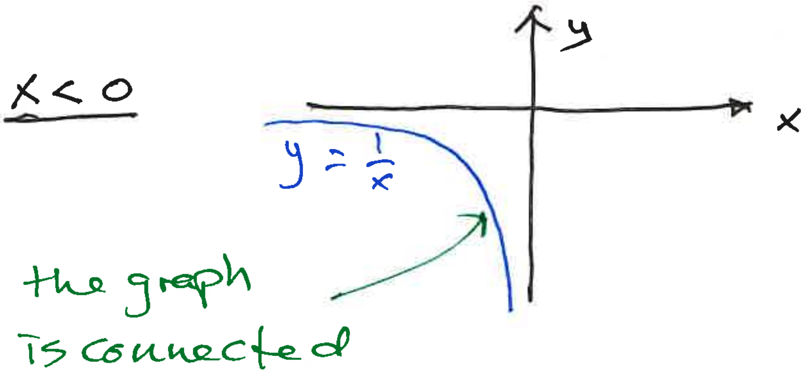
Ex $f(x) = \frac{1}{x}$ is defined for $x \neq 0$

that is, for $x \in \langle \leftarrow, 0 \rangle \cup \langle 0, \rightarrow \rangle$

for $x > 0$



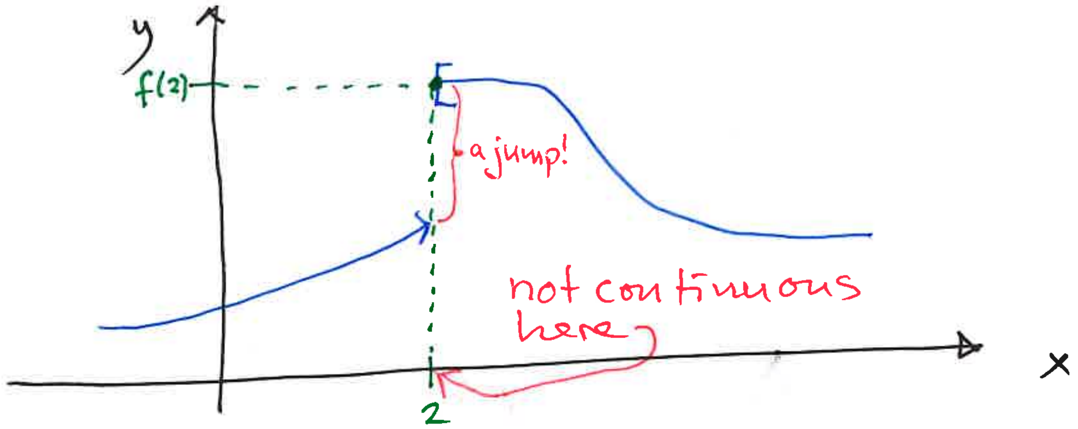
for $x < 0$



Hence $f(x) = \frac{1}{x}$ is continuous.

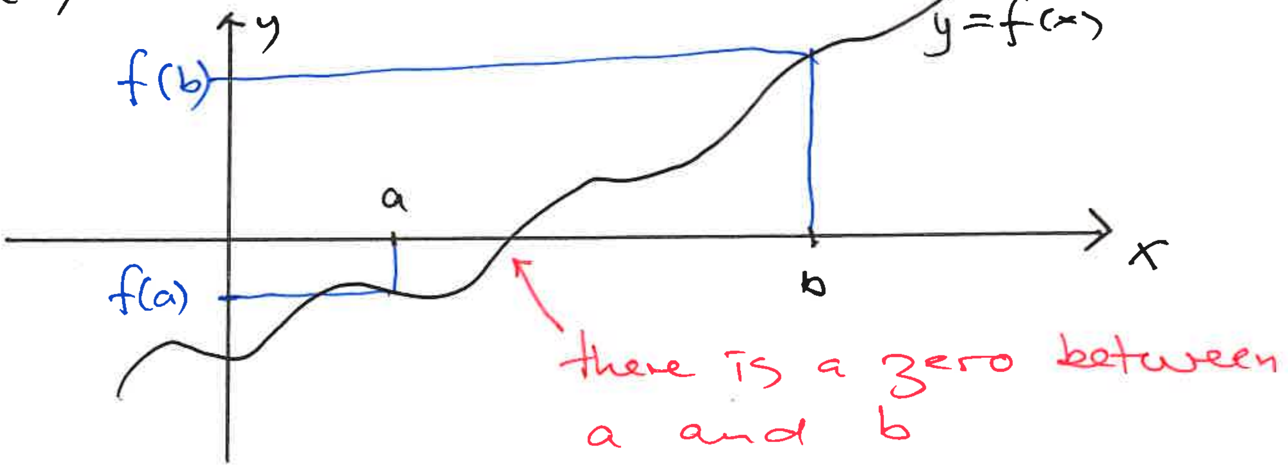
Fact All "ordinary" functions are continuous.

If the graph of $f(x)$ "jumps" then $f(x)$ is not continuous



The intermediate value theorem

If $f(x)$ is continuous in an interval I and a and b are two numbers in I with $f(a) < 0$ and $f(b) > 0$ then $f(x)$ has a root (zero) between a and b.



- in fact <u>all</u> y-values between $f(a)$ and $f(b)$ are attained for x values between a and b.

<u>Ex</u> $f(x) = x \cdot \sqrt{2x+5} - \frac{10}{x}$ has a zero between $x=1$ and $x=10$ because

- $f(1) = 1 \cdot \sqrt{2 \cdot 1 + 5} - \frac{10}{1} = \sqrt{7} - 10 < 0$
- $f(10) = 10 \cdot \sqrt{2 \cdot 10 + 5} - \frac{10}{10} = 10 \cdot 5 - 1 > 0$
- $f(x)$ is continuous for all $x > 0$.

Then the intermediate value theorem says that there is a zero between 1 and 10.