EBA2911, lecture 16, 24 Nov. 2021, Runar Ile

Plan 1. Elasticity
2. Linear approximation
3. Taylor polynomials

1. Elasticity

p = price/unit

$D(p)$ = demand of commodity with price p
= # sold units

The problem of comparing different units.

Ex A barrel of North Sea crude oil costs \$ 82.52
A litre of —— " —————— NOK 4.62

The price elasticity of the demand is

$$\varepsilon = \frac{\text{relative change in demand}}{\text{relative change in price}}$$

← numbers independ. of the choice of units

Ex In a month the price of a commodity drops from 12 thousand to 10 thousand and the demand increases from 50 mill to 60 mill. Then

$$\varepsilon = \frac{\left(\frac{60-50}{50}\right)}{\left(\frac{10-12}{12}\right)} = \frac{\left(\frac{10}{50}\right)}{\left(\frac{-2}{12}\right)} = \frac{120}{-100} = -1.2$$

Interpretation If the price increases by 1 % then the demand falls by 1.2 %.

Suppose the price is changed from p to $p+h$
Then the relative change in price is

$$\frac{p+h-p}{p} = \frac{h}{p}$$

$$\frac{\text{relative change in demand}}{\text{relative price change}}$$

$$= \frac{\left(\frac{D(p+h)-D(p)}{D(p)}\right)}{\left(\frac{h}{p}\right)} \qquad \Big| \cdot \frac{p \cdot D(p)}{p \cdot D(p)}$$

$$= \frac{D(p+h) - D(p)}{h} \cdot \frac{p}{D(p)}$$

$$\Big\downarrow h \to 0 \qquad \text{(the price change approaches 0)}$$

$$\varepsilon(p) = D'(p) \cdot \frac{p}{D(p)}$$

This is the momentary price elasticity of the demand function.

Important question

Is the revenue going up or down if we increase the price a little?

Revenue $R(p) = p \cdot D(p)$

Then the marginal revenue w.r.t. price is

$$R'(p) \overset{\text{prod.r.}}{=} 1 \cdot D(p) + p \cdot D'(p)$$

$$= D(p) \left[1 + \frac{p \cdot D'(p)}{D(p)} \right]$$

$$= \underbrace{D(p)}_{\text{always positive}} \cdot \underbrace{\left[1 + \varepsilon(p) \right]}_{\text{positive or negative ?}}$$

If $\varepsilon(p) < -1$
we get neg. $R'(p)$
so $R(p)$ is decreasing

Say: elastic demand

If $\varepsilon(p) > -1$
we get pos. $R'(p)$
so $R(p)$ is increasing

Say: inelastic demand

If $\varepsilon(p) = -1$
the demand is unit elastic

Ex $D(p) = 50 - p$ for $0 < p < 50$

Then $D'(p) = -1$ and $\varepsilon(p) = \frac{D'(p) \cdot p}{D(p)} = \frac{(-1) \cdot p}{50 - p}$

$$= \frac{-p}{50 - p}$$

<u>Question</u> In what price range do we have elastic demand ?

Have to solve the inequality

$$\varepsilon(p) < -1$$

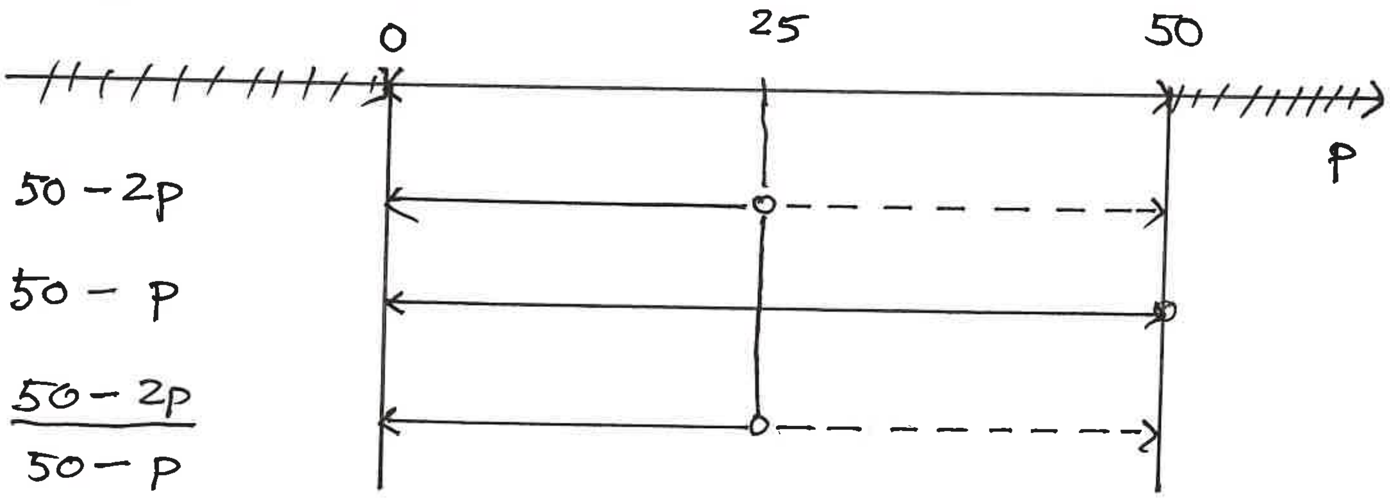
that is $\quad \dfrac{-p}{50-p} < -1 \qquad\qquad 1+1$

$$\frac{-p}{50-p} + 1 < 0$$

so $\quad \dfrac{-p+50-p}{50-p} < 0$

and get $\quad \dfrac{50-2p}{50-p} < 0$

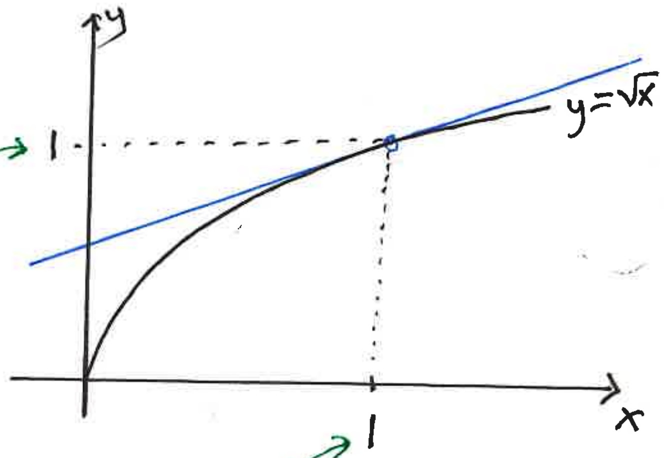
<u>Sign diagram</u>



So elastic demand for p in $\langle 25, 50)$
and inelastic demand for p in $\langle 0, 25)$
and unit elastic demand for $p = 25$.

Start : 10.59

2. Linear approximation



Ex $f(x) = \sqrt{x}$

The linear approximation of $f(x)$ <u>about</u> $x = 1$

We can find the expression for the tangent line by the point-slope formula

$$y - 1 = f'(1)\cdot(x-1)$$

so $y - 1 = \frac{1}{2}(x-1)$

or $y = \overset{=f(1)}{1} + \overset{=f'(1)}{\frac{1}{2}}(x-1)$

$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}\cdot x^{\frac{1}{2}-1}$

$= \frac{1}{2}\cdot x^{-\frac{1}{2}}$

$= \frac{1}{2}\cdot\frac{1}{x^{\frac{1}{2}}}$

$= \frac{1}{2\sqrt{x}}$

so $f'(1) = \frac{1}{2\cdot\sqrt{1}} = \frac{1}{2}$

This is a new function of x denoted

$$P_1(x) = 1 + \frac{1}{2}(x-1)$$

It is called the degree 1 Taylor polynomial of $\sqrt{x}$ about 1.

Ex $P_1(1.1) = 1 + \frac{1}{2}\cdot(1.1-1) = 1.05$

check: $\sqrt{1.1} = 1.04881\ldots$

3. Taylor polynomials

Ex $f(x) = \sqrt{x}$

The Taylor polynomial of degree 2 to $\sqrt{x}$ about $x = 1$ is

$$P_2(x) = \overbrace{f(1) + f'(1)\cdot(x-1)}^{P_1(x)} + \frac{f''(1)}{2}\cdot(x-1)^2$$

$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2$$

$f'(x) = \frac{1}{2}\cdot x^{-\frac{1}{2}}$

$f''(x) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot x^{-\frac{3}{2}}$

$= -\frac{1}{4}\cdot\frac{1}{x^{3/2}}$

$= -\frac{1}{4x\sqrt{x}}$

So $f''(1) =$

$= -\frac{1}{4\cdot 1\cdot\sqrt{1}} = -\frac{1}{4}$

Pattern

$$\boxed{P_2(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2}$$

– so $a = 1$ in the example

$\sqrt{2} = f(2) \approx P_2(2) = 1 + \frac{1}{2}(2-1) - \frac{1}{8}(2-1)^2$

$= 1 + \frac{1}{2} - \frac{1}{8} \quad = 1.375$

– an approximation to $\sqrt{2}$

(check $\sqrt{2} \simeq 1.41421\ldots$)

$P_2(1.2) = 1 + \frac{1}{2}(1.2-1) - \frac{1}{8}\cdot(1.2-1)^2$

$= 1 + 0.1 - 0.005 = 1.0950$

(check $\sqrt{1.2} = 1.0954\ldots$)

Ex $f(x) = \sqrt{x}$ about $x = 1$

Then the Taylor polynomial of degree 3 to $f(x)$ about 1 is:

$$P_3(x) = \underbrace{P_2(x)}_{\text{have already done this one!}} + \frac{f'''(1)}{6}\cdot(x-1)^3$$

$f''(x) = -\frac{1}{4x\sqrt{x}}$

$= -\frac{1}{4}\cdot x^{-\frac{3}{2}}$

$f'''(x) = \left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)\cdot x^{-5/2}$

$= \frac{3}{8x^2\sqrt{x}}$

$f'''(1) = \frac{3}{8}$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{\left(\frac{3}{8}\right)}{6}(x-1)^3$$

$$= \underbrace{1 + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2}_{P_2(x)} + \frac{1}{16}(x-1)^3$$

$$P_3(1.2) = 1 + \frac{1}{2}(1.2-1) - \frac{1}{4}(1.2-1)^2 + \frac{1}{16}\cdot(1.2-1)^3$$

$$= 1.0955$$

Pattern

$$P_3(x) = \underbrace{\overbrace{f(a) + f'(a)\cdot(x-a)}^{P_1(x)} + \frac{f''(a)}{2}\cdot(x-a)^2}_{P_2(x)} + \frac{f'''(a)}{6}\cdot(x-a)^3$$

The degree n Taylor polynomial for $f(x)$ about $x=a$ is

$$\boxed{\begin{array}{l} P_n(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n \\ \text{where } n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1 \end{array}}$$