EBA 2911, lecture 14, 11 Nov. 2021, Runar Ile

Plan
1. Implicit differentiation
2. The second derivative and curvature
3. Convex optimisation

1. Implicit differentiation

Ex 1 $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

– usual differentiation

Instead Put $y = f(x)$, so $y = \frac{1}{x}$ $\quad | \cdot x$

and get the eq. $xy = 1$

Differentiate each side w.r.t. x

and think about y as a function of x.

$$(x \cdot y)'_x = (1)'_x$$

product rule gives

$$(x)'_x \cdot y + x \cdot (y)'_x = 0$$

$$1 \cdot y + x \cdot y' = 0$$

We can solve this equation for y'.

$$x \cdot y' = -y \quad | : x$$

$$y' = -\frac{y}{x}$$

(Note: $y = \frac{1}{x}$, so $y' = -\frac{\left(\frac{1}{x}\right)}{x} = -\frac{1}{x^2}$)

This is called implicit differentiation.

can use this to find slopes of tangents to the curve.

E.g. if $x = 2$ then $xy = 1$ gives $2y = 1$

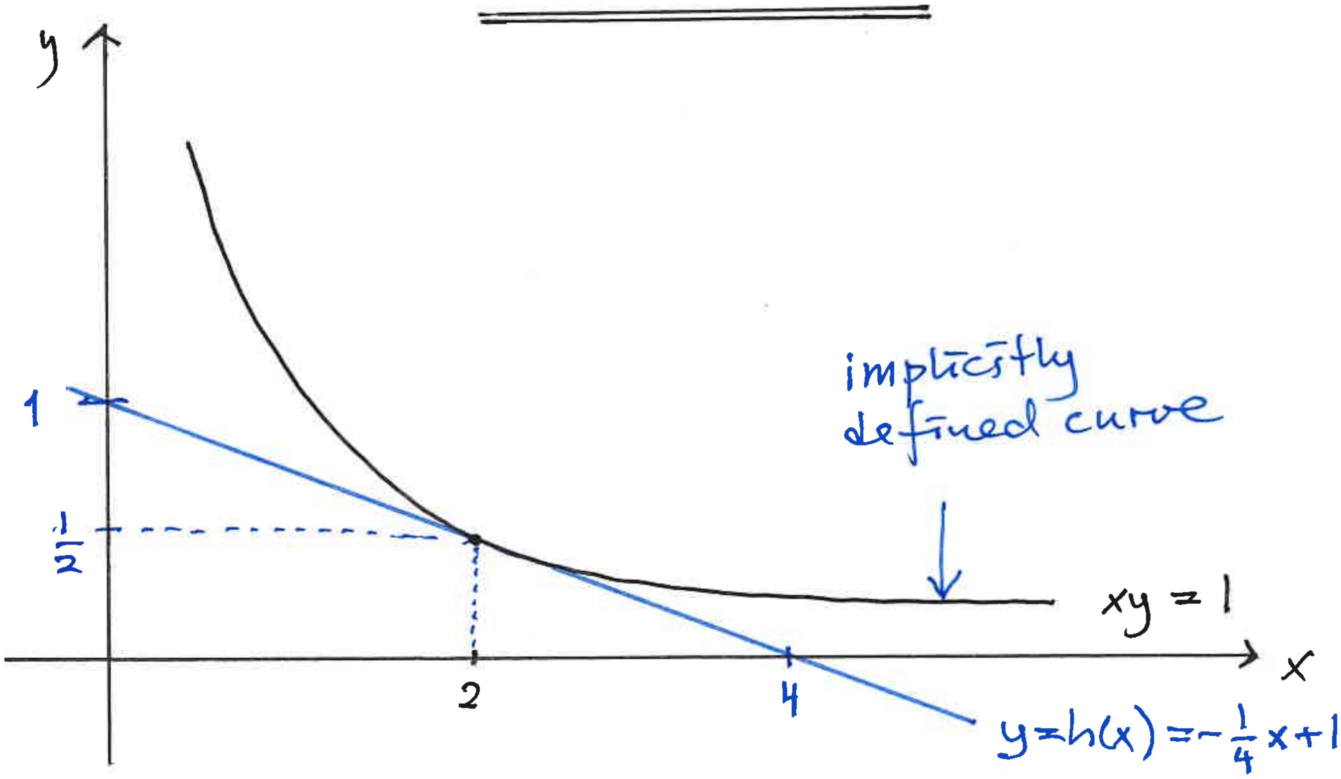
so $y = \frac{1}{2}$

Also $y'\Big|_{\substack{x=2 \\ y=\frac{1}{2}}} = -\dfrac{\frac{1}{2}}{2} = -\dfrac{1}{4}$

Possible application Find the function expression $h(x)$ for the tangent at the point $(2, \frac{1}{2})$ by the point-slope formula:

$$h(x) - \frac{1}{2} = -\frac{1}{4} \cdot (x - 2)$$

so $h(x) = -\frac{1}{4} \cdot x + 1$



Ex 2 A curve is implicitly defined by the equation

$$y^2 - x^3 = 1$$

a) Express y' by x and y using implicit differentiation.

b) Find all solutions for y when $x = 2$

c) Compute y' for these points.

Solution

a) $$(y^2)'_x - (x^3)'_x = (1)'_x$$

Chain rule $u = y$ and $g(u) = u^2$

$u'_x = y'_x$ $g'(u) = 2u$

$= 2y$

$$2y \cdot y' - 3x^2 = 0$$

and solve for y'

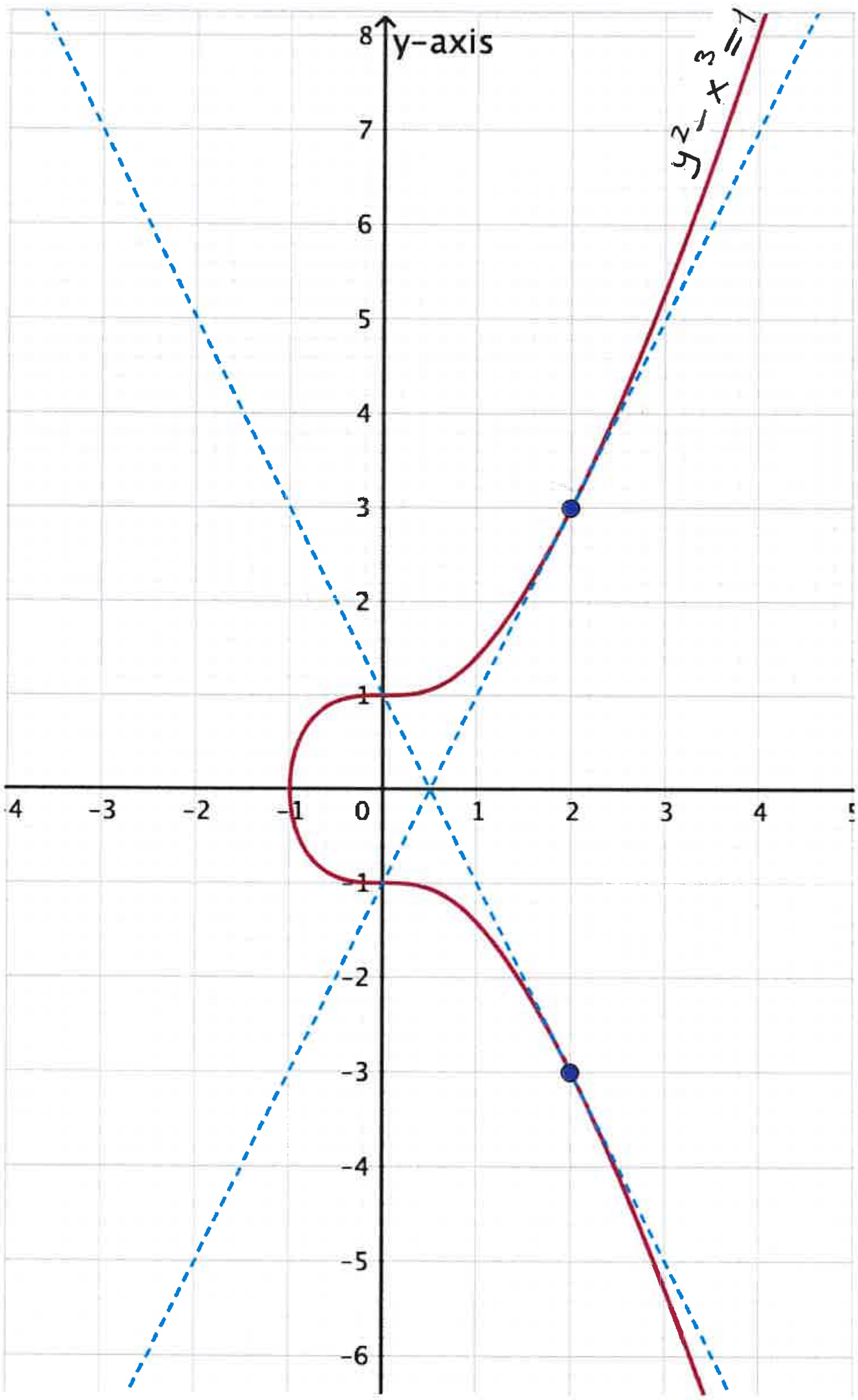
$$2y \cdot y' = 3x^2$$

$$y' = \frac{3x^2}{2y}$$

b) $x = 2$ solve $y^2 - 2^3 = 1$ for y

$$y^2 = 1 + 8 = 9$$

$$y = \pm 3$$



y-axis
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-4
-3
-2
-1
0
1
2
3
4
$y^2 - x^3 = 1$

c) $(2, -3)$: $\quad y' = \dfrac{3 \cdot 2^2}{2 \cdot (-3)} = \underline{\underline{-2}}$

$(2, 3)$: $\quad y' = \dfrac{3 \cdot 2^2}{2 \cdot 3} = \underline{\underline{2}}$

- Show graph.

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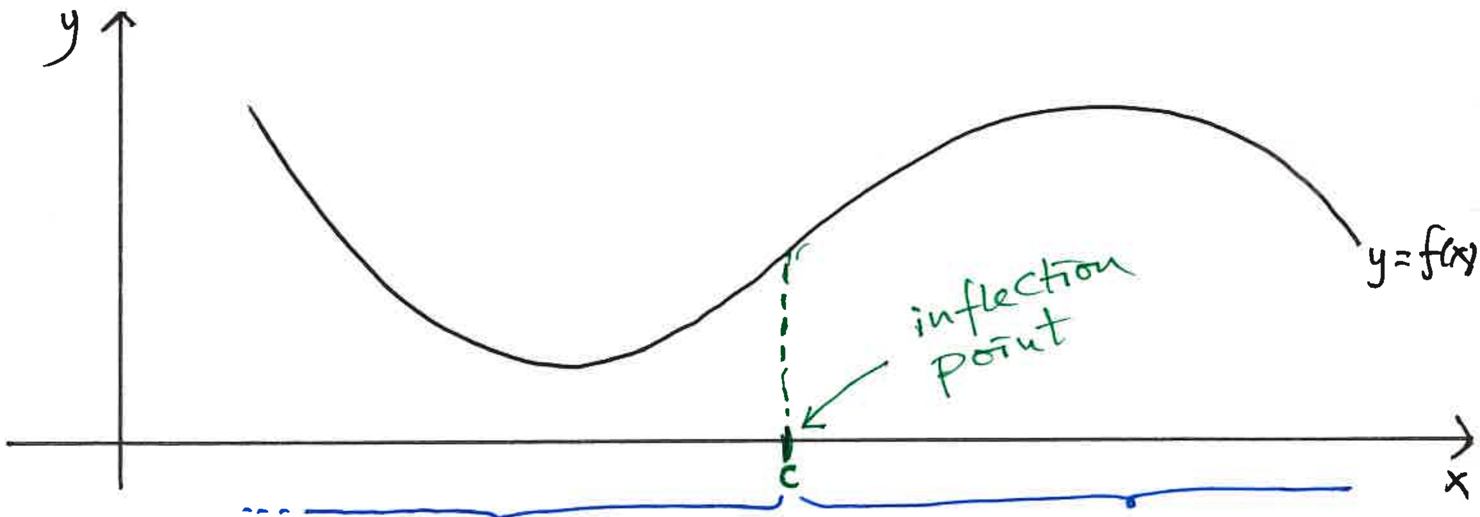
2. The second order derivative and curvature

<u>bending up</u>

graphs of <u>convex</u> functions

<u>bending down</u>

graphs of <u>concave</u> functions



f(x) is convex $\qquad$ f(x) is concave

<u>Definition</u> $f(x)$ is <u>convex</u> in the interval $[a, b]$ if $f''(x) \geq 0$ for all x i $\langle a, b \rangle$.

$f(x)$ is <u>concave</u> if $f''(x) \leq 0$ —— " ——

A number c is an <u>inflection point</u> for $f(x)$ if $f''(x)$ changes sign at $x = c$.

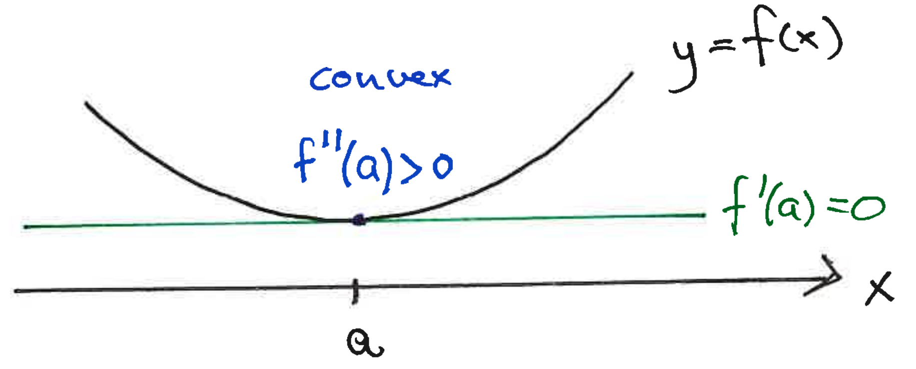
Note If $f(x)$ is convex, then $f'(x)$ is an increasing function

If $f(x)$ is concave, then $f'(x)$ is an decreasing function.

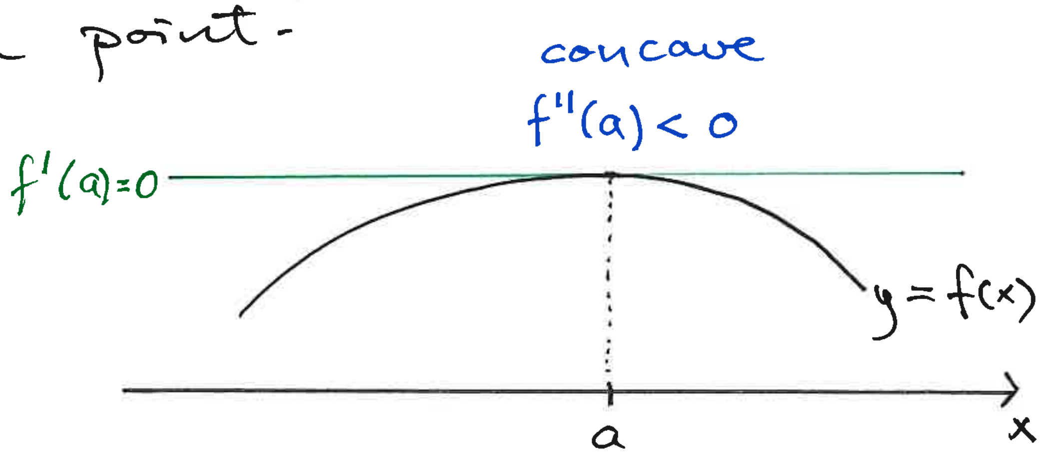
Second derivative test (sec. 8.5)

Suppose $x = a$ is a stationary point for $f(x)$.

If $f''(a) > 0$ then $x = a$ is a (local) minimum point.



If $f''(a) < 0$ then $x = a$ is a (local) maximum point.



EX $f(x) = x^3 - 3x^2 + 5$. Find local max/min points.

Solution: $f'(x) = 3x^2 - 6x$ and find stationary points by solving $f'(x) = 0$

that is $3x^2 - 6x = 0$

$3x$ is a common factor

$3x(x-2) = 0$

either $\underline{x=0}$ or $\underline{x=2}$.

Use the second derivative test to determine if they are local max. or min.

$f''(x) = [f'(x)]' = [3x^2-6x]' = 6x-6$

$f''(0) = 6\cdot 0 - 6 = -6 < 0$

so $x=0$ is a (local) maximum point.

$f''(2) = 6\cdot 2 - 6 = 6 > 0$

so $x=2$ is a (local) minimum point.

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3. Convex optimisation

Fact If $f(x)$ is convex everywhere in its domain, then any stationary point is a global minimum point

If $f(x)$ is concave everywhere: - stationary points are global max. points.

Ex $f(x) = x^4 + 5x^2 + 3$, $D_f = \langle \leftarrow, \rightarrow \rangle$

(*) Find the stationary points.

(*) Determine if they are glob. max. or min.

(*) — " — the extremal values.

<u>Solution</u> Calculate $f'(x) = 4x^3 + 10x$

(*) Stationary points: solutions to eq $f'(x) = 0$

that is $4x^3 + 10x = 0$

$x(4x^2 + 10) = 0$ a product equal to 0

only solution: $\underline{\underline{x = 0}}$.

(*) Calculate $f''(x) = 12x^2 + 10$

which is greater or equal to 10 for all x!

So $x = 0$ is a global min. point!

(*) Moreover, $f(0) = \underline{\underline{3}}$ is the global min. value.