EBA2911 Mathematics for Business Analytics autumn 2020

Exercises

I came to the position that mathematical analysis is not one of the many ways of doing economic theory: it is the only way.

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Lecture 13

on Wednesday 28 Oct. 10-11.45 in B2-060

Sec. 6.3, 6.10-11, 8.1-2, 8.4, 8.6: Rules for differentiation. Optimisation (one variable).

Here are recommended exercises from the textbook [SHSC].

Section **6.10** exercise 1, 4, 5
Section **6.11** exercise 1-3, 6, 7
Section **8.1** exercise 1
Section **8.2** exercise 5-7
Section **8.4** exercise 1-3
Section **8.6** exercise 2, 4

Problems for the exercise session
Wednesday 28 Oct. at 12-15 in CU1-067 or on Zoom

Problem 1 Make a sketch of the graphs of **two** different functions $f(x)$ with the given data. Note: You are not supposed to find any algebraic expression!
a) $f'(x)$ is negative for $x < 5$ and positive for $x > 5$
b) $f'(x)$ is positive for $x < 10$, negative for $10 < x < 15$ and positive for $x > 15$
c) $f'(x)$ is negative for $x < 5$, $f'(5) = 0$, $f'(x)$ is negative for $5 < x < 12$ and $f'(x)$ is positive for $x > 12$

Problem 2 I figure 1 you see the graph of $f'(x)$.

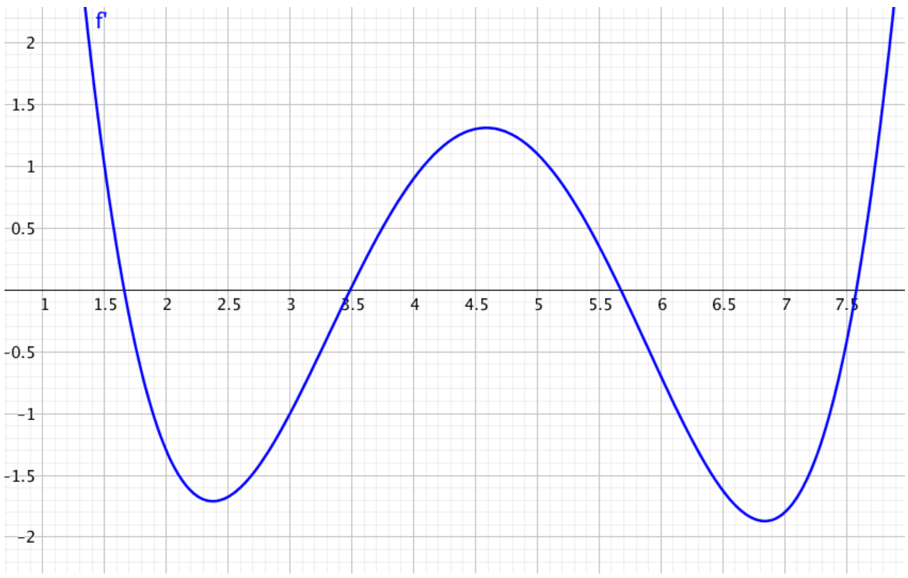

Figure 1: The graph of $f'(x)$

Determine if the statement is true or false.

a) $f'(3) < f'(4)$

b) $f(2) < f(3)$

c) $f(4.5) > f(5)$

d) $f(x)$ has a (local) minimum for $x = 3.5$

e) $f(x)$ has a (local) minimum for $2 < x < 3$

f) the graph of $f(x)$ has no local minimum points

g) $f(x)$ decreases in the interval $[6, 7]$

h) $f(x)$ increases faster around $x = 1.5$ than around $x = 5.5$

i) The derivative of $f'(x)$ is positive for $x = 7.6$

j) $f(x)$ has three stationary points

k) We cannot use the graph of $f'(x)$ to determine if $f(4.5)$ is positive

Problem 3 In figure 2 you see the graphs of $f(x)$ and $f'(x)$ in the same coordinate system. Determine which is the graph of $f(x)$ and which is the graph of $f'(x)$ in (a-c).

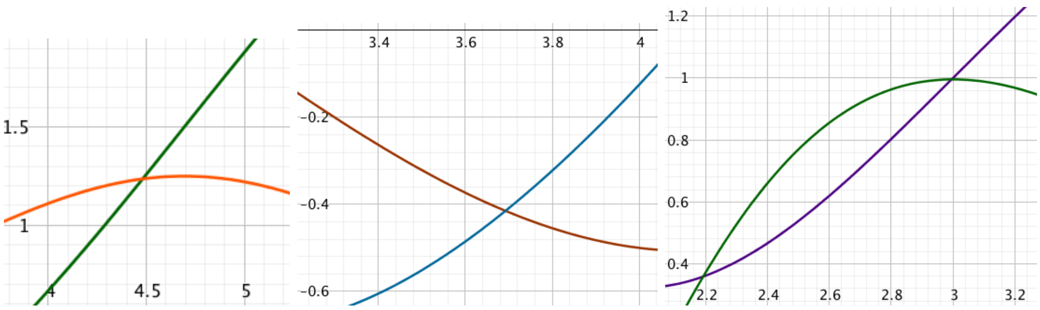

Figure 2: (a-c): The graphs of $f(x)$ and $f'(x)$

Problem 4 Determine the stationary points of $f(x)$, where $f(x)$ is strictly decreasing/increasing, and find (local) maximum and minimum points.

a) $f'(x) = 4(x+1)(x-2)(x-5)$

b) $f'(x) = (x-20)e^x$

c) $f'(x) = \frac{(3x-5)(10-2x)}{x^2-6x+10}$

d) $f'(x) = \ln(x) - 1.12$

e) $f'(x) = \ln(x^2 - 6x + 10)$

f) $f'(x) = \ln(x^2 - 8)$, $(x > 2.9)$

g) $f'(x) = e^{2x} - 4e^x + 3$

h) $f'(x) = e^{x^2-3} - 2$

Problem 5 Determine maximum and minimum for these functions.

a) $f(x) = 1000 - 0.2x$ and $D_f = [50, 250]$

b) $f(x) = 0.2x^2 - 2.8x + 19.8$ and $D_f = [2, 12]$

c) $f(x) = 20 - \frac{1}{x-5}$ and $D_f = [6, 15]$

d) $f(x) = 10xe^{-0.1x}$ and $D_f = [2, 30]$

e) $f(x) = 2x^3 - 33x^2 + 168x + 9$ and $D_f = [2.5, 8.6]$

f) $f(x) = \ln(1 + e^{-x})$ and $D_f = [4, 5]$

Problem 6 The mean value theorem says that a function $f(x)$ which is defined and continuous (connected graph) in the interval $[a, b]$ and is differentiable (no cusps) then there is a number c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

a) We have $f(x) = \sqrt{\ln[(x-4)^2] + 5} + x^3 - 4x$. Calculate $\frac{f(6)-f(2)}{4}$ and explain why there is a number c with $2 < c < 6$ such that $f'(c) = 48$.

b) We have a continuous and differentiable function $f(x)$ with $f(13) = 600e^{1.14} = f(17)$. Explain why $f(x)$ has a stationary point between 13 and 17.

Problem 7 Compute the expression for the derivative of $f(x)$.

a) $f(x) = \ln(x^2 - 7x + 13)$ b) $f(x) = e^{0.035x^2}$ c) $f(x) = \sqrt{e^{2x} + 4x + 5}$ d) $f(x) = \dfrac{x}{\ln(1-x)}$

Problem 8 (Multiple choice exam spring 2016, problem 12)
We have the function $f(x) = \ln(x^2 + 4x + 5)$. Which statement is true?

(A) The function f is increasing in $[2, \rightarrow\rangle$
(B) The function f is increasing in $[-2, \rightarrow\rangle$
(C) The function f is increasing in $\langle\leftarrow, 2]$
(D) The function f is increasing in $\langle\leftarrow, -2]$
(E) I choose not to solve this problem.

Problem 9 (Multiple choice exam autumn 2016, problem 10)
We have the function $f(x) = \frac{x^2-3x}{x+1}$. Which statement is true?

(A) The function f has no local minimum points
(B) The function f has one local minimum point, and it is $x = -3$
(C) The function f has one local minimum point, and it is $x = 1$
(D) The function f has several local minimum points
(E) I choose not to solve this problem.

Problem 10 (Multiple choice exam spring 2018, problem 10)
We have the function $f(x) = x^2 e^{1-x}$. Which statement is true?

(A) The function f has one local maximum point $x = a$ with $a > 0$
(B) The function f has several local maximum points
(C) The function f has one local maximum point $x = 0$
(D) The function f has one local maximum point $x = a$ with $a < 0$
(E) I choose not to solve this problem.

Answers

Problem 1
There are many possibilities. Compare with other students, ask the learning assistants!

Problem 2

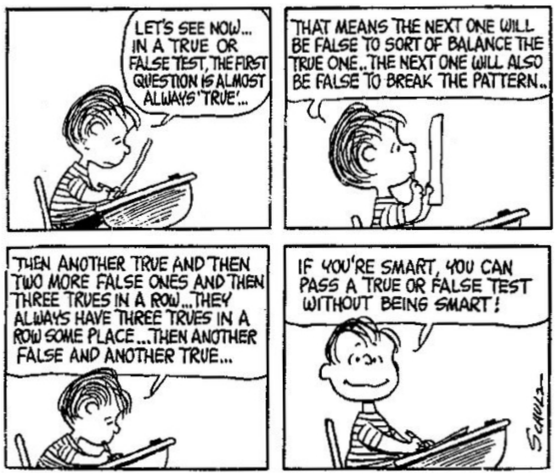

Figure 3: True or false

Problem 3
a) $f(x)$: Green b) $f(x)$: Brown c) $f(x)$: Violet

Problem 4

a) Stationary points: $x = -1$, $x = 2$, $x = 5$. $f(x)$ is strictly decreasing for $x \leqslant -1$, $f(x)$ is strictly increasing for $-1 \leqslant x \leqslant 2$, $f(x)$ is strictly decreasing for $2 \leqslant x \leqslant 5$, $f(x)$ is strictly increasing for $x \geqslant 5$. Hence $x = -1$ is a local minimum point, $x = 2$ is a local maximum point and $x = 5$ is a local minimum point.

b) Stationary points: Only $x = 20$. $f(x)$ is strictly decreasing for $x \leqslant 20$ and $f(x)$ is strictly increasing for $x \geqslant 20$. Hence $x = 20$ is a global minimum point.
c) Stationary points: $x = \frac{5}{3}$ and $x = 5$. $f(x)$ is strictly decreasing for $x \leqslant \frac{5}{3}$, $f(x)$ is strictly increasing for $\frac{5}{3} \leqslant x \leqslant 5$, $f(x)$ is strictly decreasing for $x \geqslant 5$. Hence $x = \frac{5}{3}$ is a local minimum point and $x = 5$ is a local maximum point.
d) Stationary points: Only $x = e^{1.12}$. $f(x)$ is strictly decreasing for $x \leqslant e^{1.12}$ and $f(x)$ is strictly increasing for $x \geqslant e^{1.12}$. Hence $x = e^{1.12}$ is a global minimum point.
e) Stationary points: Only $x = 3$. $f(x)$ is strictly increasing for all x. Hence $x = 3$ is nether a local minimum point nor a local maximum point (a *terrace point*).
f) Stationary points: Only $x = 3$. $f(x)$ is strictly decreasing for $2.9 < x \leqslant 3$, $f(x)$ is strictly increasing for $x \geqslant 3$. Hence $x = 3$ is a global minimum point.
g) $f'(x) = (e^x - 1)(e^x - 3)$. Stationary points: $x = 0$ and $x = \ln(3)$. $f(x)$ is strictly increasing for $x \leqslant 0$, $f(x)$ is strictly decreasing for $0 \leqslant x \leqslant \ln(3)$ and $f(x)$ is strictly increasing for $x \geqslant \ln(3)$. Hence $x = 0$ is a local maximum point and $x = \ln(3)$ is a local minimum point.
h) Stationary points: $x = \pm\sqrt{3 + \ln(2)}$. $f(x)$ is strictly increasing for $x \leqslant -\sqrt{3 + \ln(2)}$, $f(x)$ is strictly decreasing for $-\sqrt{3 + \ln(2)} \leqslant x \leqslant \sqrt{3 + \ln(2)}$ and $f(x)$ is strictly increasing for $x \geqslant \sqrt{3 + \ln(2)}$. Hence $x = -\sqrt{3 + \ln(2)}$ is a local maximum point and $x = \sqrt{3 + \ln(2)}$ is a local minimum point.

Problem 5 We use the extreme value theorem (Sec. 8.4, Thm. 8.4.1, p. 294).
a) min $f(250) = 950$ max: $f(50) = 990$
b) min $f(7) = 10$ max: $f(2) = 15 = f(12)$
c) min: $f(6) = 19$ max: $f(15) = 19.9$
d) min: $f(30) = 14.94$ max: $f(10) = 36.79$
e) min: $f(7) = 254 = f(2.5)$ max: $f(8.6) = 285.23$
f) min: $f(5) = 0.00672$ max: $f(4) = 0.01815$

Problem 6
a) $\frac{f(6)-f(2)}{4} = 48$. Because $f(x)$ is continuous and differentiable for all x the mean value theorem (see also Sec. 8.4, Thm. 8.4.2, p. 298) says that there is a number c with $2 < c < 6$ such that $f'(c) = 48$.
b) From the mean value theorem there is a number c in the interval $\langle 13, 17\rangle$ such that $f'(c) = 0$ and then $x = c$ is a stationary point for $f(x)$.

Problem 7

(a) $f'(x) = \dfrac{2x - 7}{x^2 - 7x + 13}$ (b) $f'(x) = 0.07xe^{0.035x^2}$

(c) $f'(x) = \dfrac{e^{2x} + 2}{\sqrt{e^{2x} + 4x + 5}}$ (d) $f'(x) = \dfrac{(1-x)\ln(1-x) + x}{(1-x)[\ln(1-x)]^2}$

Problem 8
B

Problem 9
C

Problem 10
A