EBA 2911, extra lecture A, 9. Nov 2020, Runar Ile

Plan Problems from set 14.

1d implicit differentiation

2 implicitly defined curves

6b concave/convex functions

7b ~~tangent~~ functions of inflection points

5a the graphs of f, f', f

Prob 1d $x^3 - 3xy + y^2 = 0 \qquad (*)$

We find an expression for y' in y and x by differentiating both sides with respect to x, and solve the new equation for y'.

Partial calculations

$$(xy)' \overset{\text{product.r}}{=} (x)' \cdot y + x \cdot y'$$
$$= 1 \cdot y + x \cdot y' = y + x \cdot y'$$

Chain rule : $(y^2)' = 2y \cdot y'$

Then $(*)$ gives

$$3x^2 - 3 \cdot (y + x \cdot \underline{y'}) + 2y \cdot \underline{y'} = 0$$

that is $\quad 3x^2 - 3y - 3x\underline{y'} + 2y\underline{y'} = 0$

collect y'-terms

$$(2y - 3x)\,y' = 3y - 3x^2 = 3(y - x^2)$$

So $$y' = \frac{3(y-x^2)}{2y-3x}$$

Assume $\underline{x=2}$, we find the possible y-values by solving (*) with $x=2$:

$$2^3 - 3\cdot 2\cdot y + y^2 = 0$$

$$y^2 - 6y = -8$$

$$(y-3)^2 = -8 + 3^2 = 1$$

so either $y-3=1$ or $y-3=-1$

that is $\underline{y=4}$, $\underline{y=2}$

We use the point-slope formula to find the two tangentfunctions through the points $(2,4)$ and $(2,2)$.

$\underline{(2,4)}$ $$y' = \frac{3(4-2^2)}{2\cdot 4 - 3\cdot 2} = \frac{3\cdot 0}{2} = 0$$

so the tangent function is constant: $\underline{h_1(x) = 4}$

$\underline{(2,2)}$ $$y' = \frac{3\cdot(2-2^2)}{2\cdot 2 - 3\cdot 2} = \frac{3\cdot(-2)}{(-2)} = 3$$

point-slope formula gives $h_2(x) - 2 = 3(x-2)$

that is $\underline{h_2(x) = 3x - 4}$

Prob 2 Ellimination is the strategy.

In 1a, c and d we get two y-values for one x-value. So 1a, c and d cannot be the blue graph (to the right)

So 1 b has to be the blue

(to the left) (bottom)
The red and the green are symmetric and so the tangents have to be symmetric too. In particular the tangents have slopes which are equal apart from opposite sign.
This is the case for 1a and c.
So 1d has to be the purple graph

In 1a both y-values are positive, in 1c one y-value is negative.
If the thicker grey lines are coordinate axes, then

1 a has to be the green

1 c —— " —— red

EBA2911 Mathematics for Business Analytics
autumn 2020
Exercises

I came to the position that mathematical analysis is not one of the many ways of doing economic theory: it is the only way.

R. Lucas

Lecture 14

on Wednesday 4 Nov. at 10-11.45 streaming

Sec. 7.1, 6.9, 8.6-7:

Implicit differentiation. The second order derivative, convex/concave functions.

Here are recommended exercises from the textbook [SHSC].

Section **7.1** exercise 1, 4, 6, 7a
Section **6.9** exercise 1-4
Section **8.6** exercise 1-4, 6a
Section **8.7** exercise 1-3, 5

Problems for the exercise session Wednesday 4 Nov. at 12-15 on Zoom

Problem 1 Find an expression for y' in terms of y and x by implicit differentiation. Find all solutions for y with $x = a$ and determine the expression for the tangent function in each of these points.

a) $x^2 + 25y^2 - 50y = 0$ and $a = 4$

b) $x^{3.27}y^{1.09} = 1$ and $a = 1$

c) $x^4 - x^2 + y^4 = 0$ and $a = \frac{\sqrt{2}}{2}$

d) $x^3 - 3xy + y^2 = 0$ and $a = 2$

Problem 2 in figure 1 you see the graphs of the implicit defined curves in Problem 1. Determine the curves and the equations which belong together. Also draw the tangents in Problem 1.

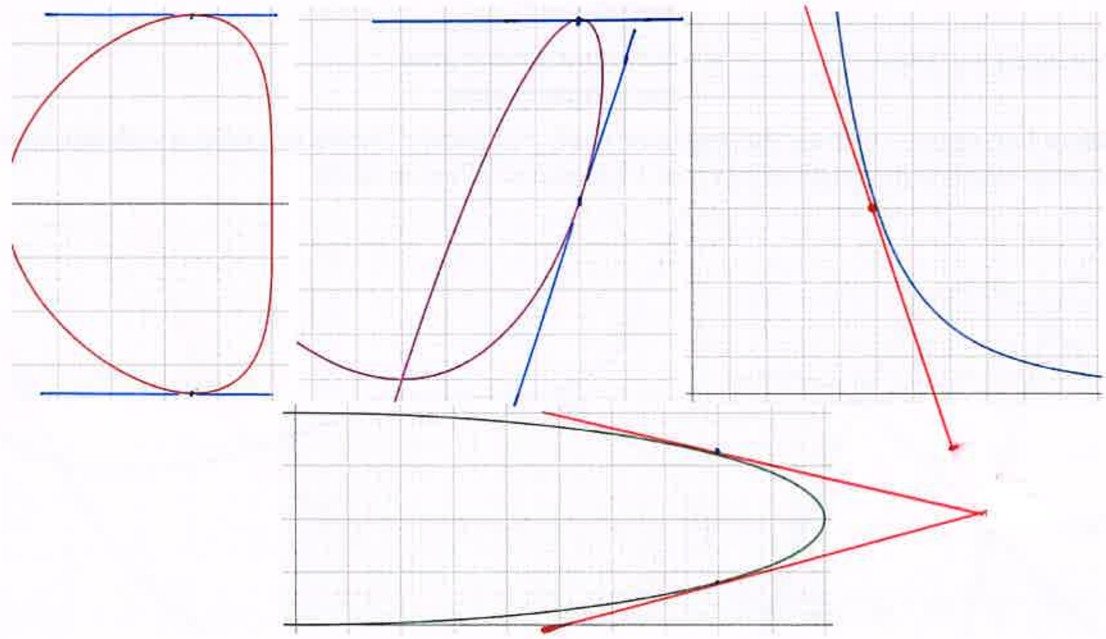


Figure 1: Four implicitly defined curves

Prob 6b $\quad f(x) = \ln(x^2-2x+2) - \frac{x}{4} + 1$

Note: $x^2-2x+2 = (x-1)^2+1 \geq 1$

so $f(x)$ is defined one the whole number line.

$$f'(x) = \left[\ln(x^2-2x+2)\right]' - \frac{1}{4} + 0$$

Chain rule with

$u = x^2-2x+2 \qquad g(u) = \ln(u)$

$u' = 2x-2 \qquad g'(u) = \frac{1}{u}$

$$= (2x-2)\cdot\frac{1}{x^2-2x+2} - \frac{1}{4}$$

$$= \frac{2x-2}{x^2-2x+2} - \frac{1}{4}$$

$$f''(x) = \frac{(2x-2)'\cdot(x^2-2x+2)-(2x-2)\cdot(x^2-2x-2)'}{(x^2-2x+2)^2} - 0$$

$$= \frac{2(x^2-2x+2)-(2x-2)(2x-2)}{(x^2-2x+2)^2}$$

$$= \frac{2x^2-4x+4-(4x^2-8x+4)}{(x^2-2x+2)^2}$$

$$= \frac{-2x^2+4x}{(x^2-2x+2)^2} = \frac{-2x(x-2)}{\left[(x+1)^2+1\right]^2}$$

Solve equation $f''(x) = 0$

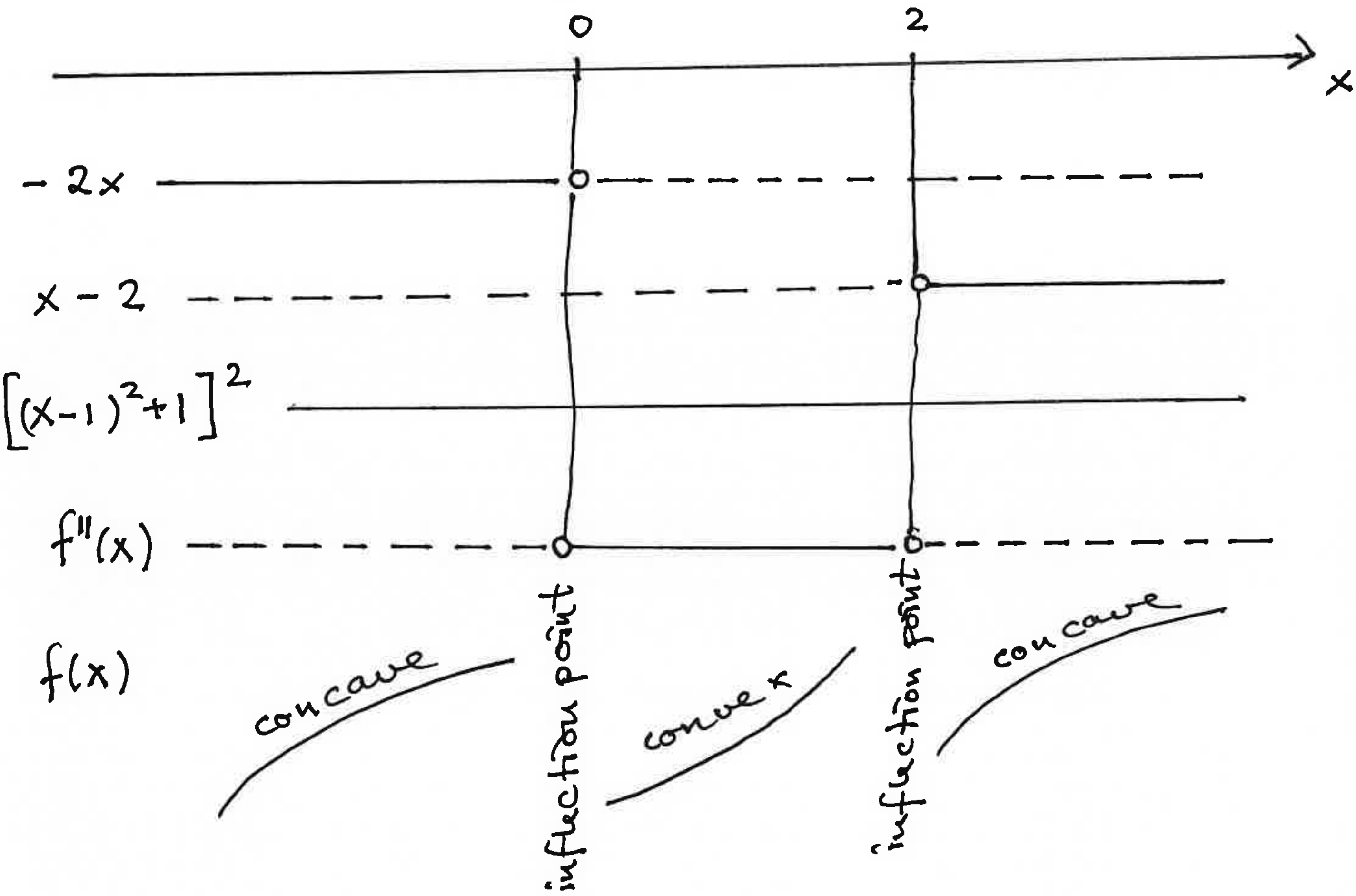
that is $-2x(x-2) = 0$

(the denominator is ≥ 1).

that is $-2x = 0$ or $x - 2 = 0$

$\underline{\underline{x = 0}}$, $\underline{\underline{x = 2}}$

The sign diagram for $f''(x)$.



Conclusion

$f(x)$ is concave for x in $\langle \leftarrow, 0]$

$f(x)$ is convex for x in $[0, 2]$

$f(x)$ is concave for x in $[2, \rightarrow\rangle$

Then the inflection points are

$\underline{\underline{x = 0}}$ and $\underline{\underline{x = 2}}$

Prob 7b We use the point-slope formula to find the expressions for the tangent functions at the inflection points.

Point-slope formula :

$$y - y_0 = a\cdot(x - x_0)$$

a = slope of the line and (x_0, y_0) is a point on the line

$$f'(0) = \frac{2\cdot 0 - 2}{0^2 - 2\cdot 0 + 2} - \frac{1}{4} = \frac{-2}{2} - \frac{1}{4} = -\frac{5}{4} = -1.25$$

$$y_0 = f(0) = \ln(0^2 - 2\cdot 0 + 2) - \frac{0}{4} + 1 = \ln(2) + 1$$

$$h_0(x) - (\ln(2)+1) = -1.25\cdot(x - 0)$$

so $$\underline{\underline{h_0(x) = -1.25x + \ln(2) + 1}}$$

$$f'(2) = \frac{2\cdot 2 - 2}{2^2 - 2\cdot 2 + 2} - \frac{1}{4} = \frac{2}{2} - \frac{1}{4} = 0.75$$

$$y_0 = f(2) = \ln(2^2 - 2\cdot 2 + 2) - \frac{2}{4} + 1 = \ln 2 + 0{,}5$$

$$h_2(x) - (\ln(2) + 0{,}5) = 0.75(x - 2)$$

$$\underline{\underline{h_2(x) = 0.75x + \ln(2) - 1}}$$

Prob. 5a The simplest is to show that something is wrong! Ex:

*) Suppose $f(x)$ is the violet one (to the left).

Then $f'(x)$ is negative for $0.6 \leq x \leq 1.3$

But none of the other graphs are negative for all x in this interval.

Hence $f(x)$ cannot be the violet one!

*) Suppose $f(x)$ is the green one (in the middle)

Then $f'(1.3) = 0$ (a local max. point), then

$f'(x)$ has to be the violet one and

$f''(x)$ has to be the red one (to the right)

- but that is not possible since the slope of the tangent of $f'(x)$ at $x = 1.3$ is negative, while $f''(1.3) = 0.6 > 0$

Hence $f(x)$ cannot be the green one!

Conclusion: $f(x)$ is the red one.

Since $f'(x)$ cannot be the violet one (as before)

$f'(x)$ is the green one and

$f''(x)$ is the violet one.