EBA 2911, lecture 9, 5 oct. 2020, Runar Ile

**Plan**
1. Rational functions and asymptotes
2. Hyperbolas
3. Continuity and the intermediate value theorem

## 1. Rational functions and asymptotes

Rational function $f(x) = \dfrac{p(x)}{q(x)}$ polynomials

**Ex** $f(x) = \dfrac{2x+1}{x^2+3}$ – would like to see what happens when $x$ is big.

divide by $x^2$ both in the numerator and in the denominator

$$= \frac{\frac{2x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}} = \frac{\frac{2}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}} \xrightarrow[x \to \pm\infty]{} \frac{0}{1} = 0$$

$$f(1000) = \frac{\frac{2}{1000} + \frac{1}{1000^2}}{1 + \frac{3}{1000^2}} = 0.00200099\ldots$$

This means that the line $y = 0$ ($x$ is free) is a horizontal asymptote for $f(x)$.

**Ex** $f(x) = \dfrac{2x+1}{(x-1)(x-5)}$ $\quad (x \neq 1,\ x \neq 5)$

If $x \longrightarrow 1^-$ " $x$ is approaching 1 from below "

$0.9,\ 0.99,\ 0.999\ \ldots$

then

$$\left.\begin{aligned} x-1 &\longrightarrow 0^- \\ x-5 &\longrightarrow -4^- \\ 2x+1 &\longrightarrow 3^- \end{aligned}\right\} \text{implies} \quad f(x) = \frac{\overbrace{(2x+1)}^{\to 3^-}}{\underbrace{(x-1)}_{\downarrow\, 0^-}\underbrace{(x-5)}_{\downarrow\, -4^-}} \xrightarrow[x \to 1^-]{} +\infty$$

Dermed vil $f(x) = \dfrac{\overbrace{(2x+1)}^{\to 3^-}}{\underbrace{(x-1)}_{\to 0^-} \cdot \underbrace{(x-5)}_{-4^-}} \xrightarrow[x \to 1^-]{} +\infty$

Hvis $x \longrightarrow 1^+$ „x nærmer seg 1 fra oversiden"

Da vil $x-1 \longrightarrow 0^+$

$x-5 \longrightarrow -4^+$

$2x+1 \longrightarrow 3^+$

Dermed vil $f(x) = \dfrac{\overbrace{(2x+1)}^{\to 3^+}}{\underbrace{(x-1)}_{\to 0^+} \cdot \underbrace{(x-5)}_{\to -4^+}} \xrightarrow[x \to 1^+]{} -\infty$

<u>Konklusjon</u> Linjen $x = 1$ ($y$ fri) er en vertikal asymptote for $f(x)$.

Tilsvarende: $f(x) \xrightarrow[x \to 5^+]{} +\infty$

og $f(x) \xrightarrow[x \to 5^-]{} -\infty$

Altså er linjen $x = 5$ ($y$ fri) en vertikal asymptote for $f(x)$.

NB: $f(x)$ har også en horisontal asymptote $y = 0$ ($x$ fri).

If $x \to 1^+$ e.g. 1.1 , 1.01 , 1.001 ....

then

$$\left.\begin{array}{l} x-1 \to 0^+ \\ x-5 \to -4^+ \\ 2x+1 \to 3^+ \end{array}\right\} \text{implies } f(x) = \frac{\overbrace{(2x+1)}^{\to 3^+}}{\underbrace{(x-1)}_{\downarrow\, 0^+}\underbrace{(x-5)}_{\downarrow\, -4^+}} \xrightarrow[x \to 1^+]{} -\infty$$

The line $x=1$ ($y$ free) is a vertical asymptote for $f(x)$.

Similarily: $f(x) \xrightarrow[x \to 5^+]{} +\infty$

$$f(x) \xrightarrow[x \to 5^-]{} -\infty$$

The line $x=5$ ($y$ free) is a vertical asymptote for $f(x)$.

<u>Note</u> $f(x)$ also has the horizontal asymptote $y = 0$ ($x$ free).

## <u>Non-vertical asymptotes</u>

<u>Ex</u> $f(x) = x - 5 + \frac{2}{x-4}$ has a vertical asymptote $x = 4$ ($y$ free)

Put $g(x) = x - 5$

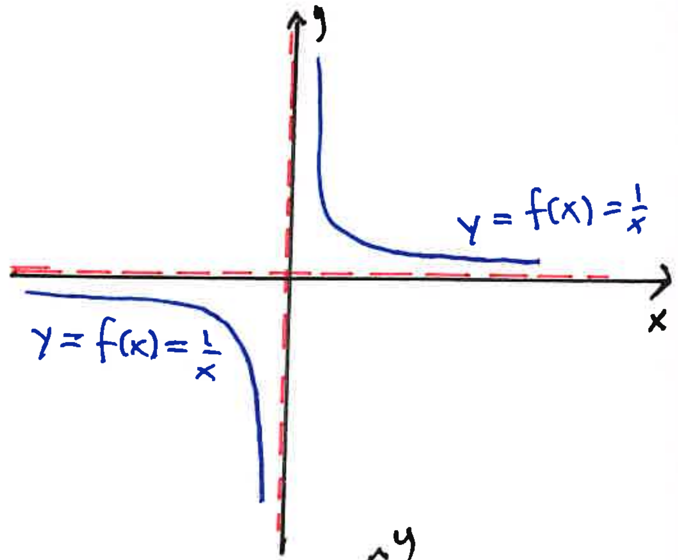
Then the graph of $g(x)$ is a non-vertical asymptote for $f(x)$ because

$$f(x) - g(x) = \frac{2}{x-4} \xrightarrow[x \to \pm\infty]{} 0$$

Note $f(x) = \dfrac{(x-5)(x-4)+2}{(x-4)} = \dfrac{x^2-9x+22}{x-4}$

## 2. Hyperbolas

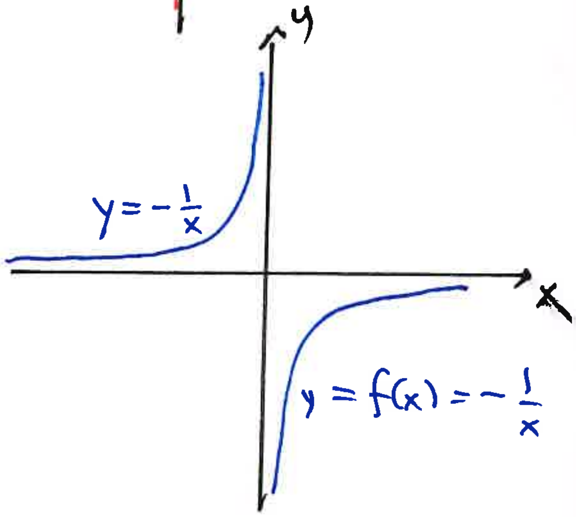
**Ex** $f(x) = \dfrac{1}{x}$ $(x \neq 0)$




The line $x=0$ is a vertical asymptote for $f(x)$

The line $y=0$ is a horizontal asymptote for $f(x)$.

**Ex** $f(x) = -\dfrac{1}{x}$ $(x \neq 0)$




Definition A function $f(x)$ is a hyperbola function if it can be written as

$$f(x) = c + \frac{a}{x-b}$$

**Ex** $f(x) = \dfrac{3x-5}{x-2}$ is a hyperbola function because polynomial division gives

$$(3x-5) : (x-2) = 3 + \frac{1}{x-2}$$
$$\underline{-(3x-6)}$$
$$1$$

$\cdot(x-2)$

so $a = 1$
$b = 2$
$c = 3$

so $f(x) = 3 + \dfrac{1}{x-2}$ $(x \neq 2)$

We have $\quad 3 + \dfrac{1}{x-2} \xrightarrow[x \to 2^-]{} -\infty$

$$3 + \frac{1}{x-2} \xrightarrow[x \to 2^+]{} +\infty$$

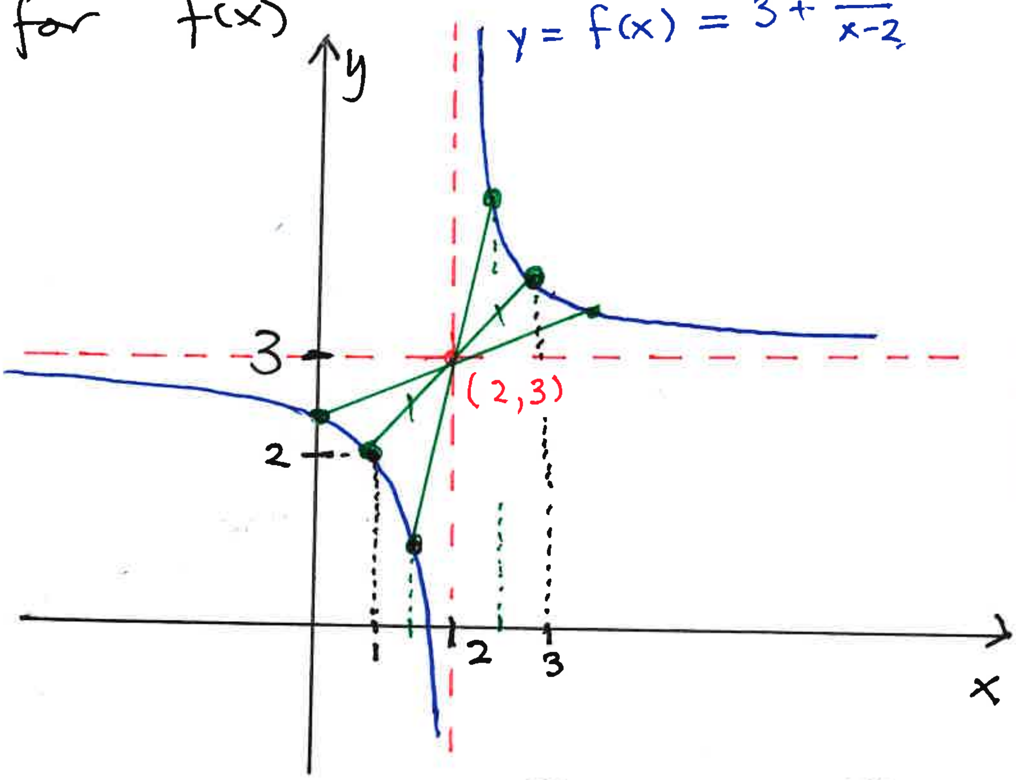
so the line $\boxed{x = 2}$ is a vertical asymptote for $f(x)$

Also $\quad 3 + \dfrac{1}{x-2} \xrightarrow[x \to \pm\infty]{} 3$

so the line $\boxed{y = 3}$ is a horizontal asymptote for $f(x)$

$f(1) = 3 + \frac{1}{1-2} = 2$

$f(3) = 3 + \frac{1}{3-2} = 4$




The graph of a hyperbola function is symmetric through the intersection point of the asymptotes.

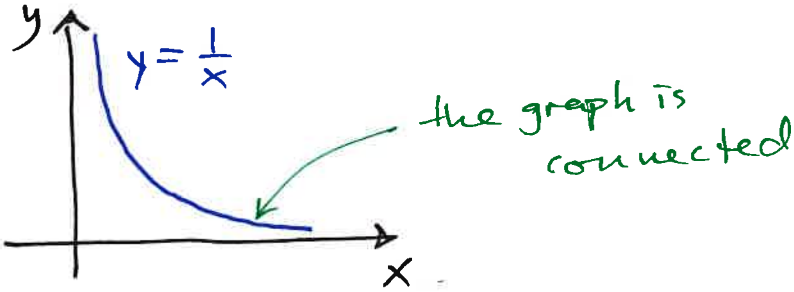
## 3. Continuity and the intermediate value theorem

A function is continuous if the graph is connected for every interval in the domain of definition.

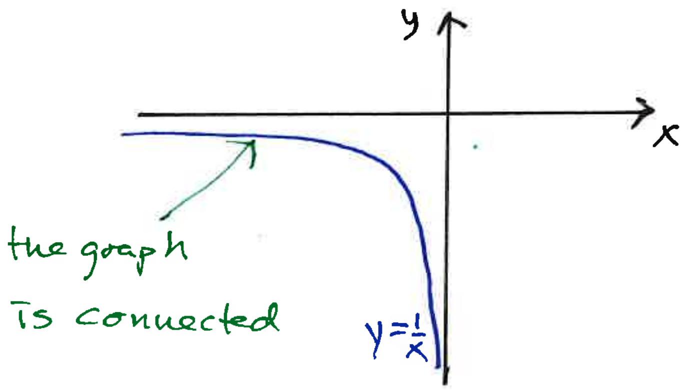
**Ex** $f(x) = \frac{1}{x}$ is defined for $x \neq 0$

that is $x \in \langle \leftarrow, 0 \rangle \cup \langle 0, \rightarrow \rangle$

For $x \in \langle 0, \rightarrow \rangle$:



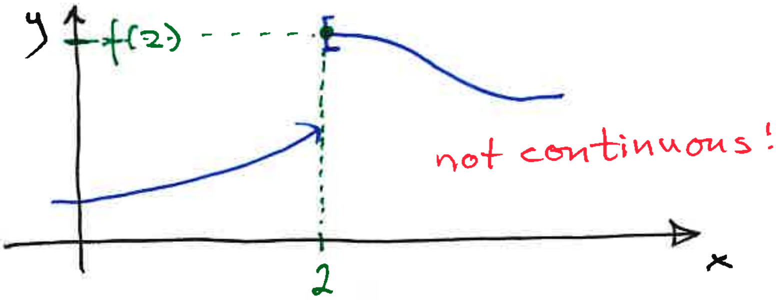

For $x \in \langle \leftarrow, 0 \rangle$:




Hence $f(x) = \frac{1}{x}$ is continuous.

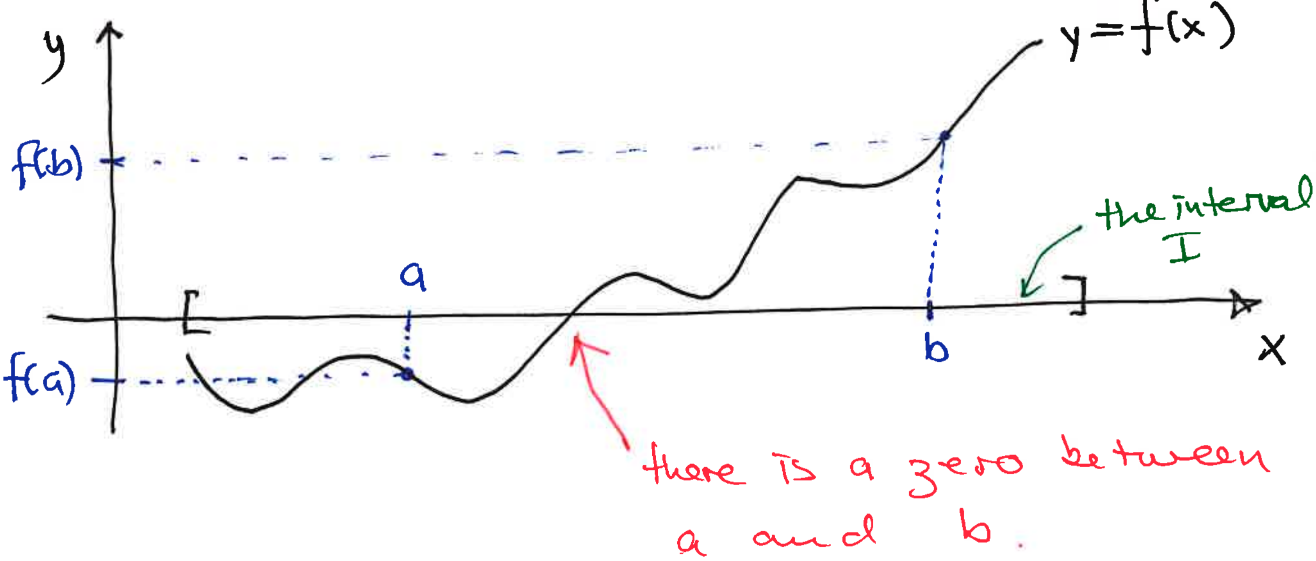
**Fact** All "ordinary" functions are continuous

If the graph of $f(x)$ "jumps" then it is not continuous.



# The intermediate value theorem

If $f(x)$ is continuous in an interval $I$ and $a$ and $b$ are two numbers in $I$ with $f(a) < 0$ and $f(b) > 0$ then there is a root (zero) for $f(x)$ between $a$ and $b$.




<u>Ex</u> $f(x) = x\cdot\sqrt{2x+5} - \frac{10}{x}$ has a zero between $x=1$ and $x=10$ because

- $f(1) = 1\cdot\sqrt{2\cdot 1+5} - \frac{10}{1} = \sqrt{7} - 10 < 0$
- $f(10) = 10\cdot\sqrt{2\cdot 10+5} - \frac{10}{10} = 10\cdot 5 - 1 > 0$
- $f(x)$ is continuous for $x > 0$

Then the intermediate value thm. says that there is a zero between 1 and 10.