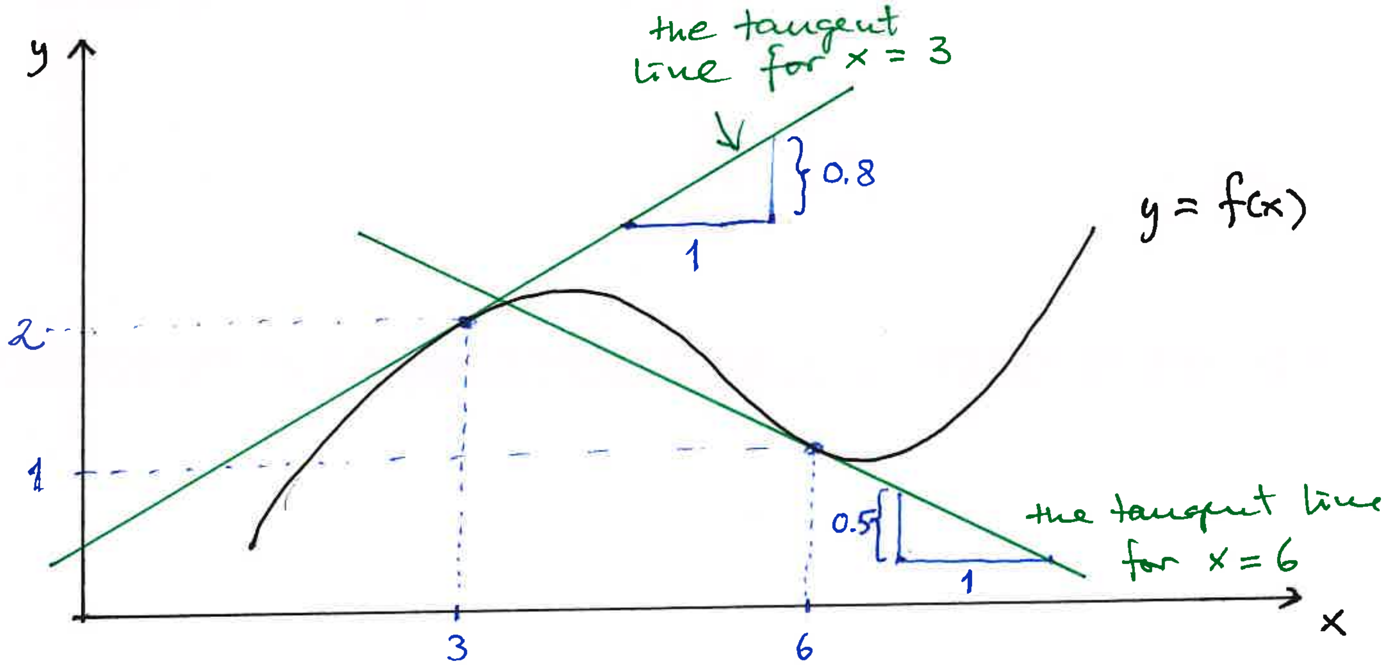
EBA 2911, lecture 11, 20 Oct. 2020, Runar Ile

Plan
1. Tangents and the derivative
2. The derivative as a function
3. Rules for differentiation



The tangent of the graph of $f(x)$ at the point $(3, 2)$ has slope 0.8

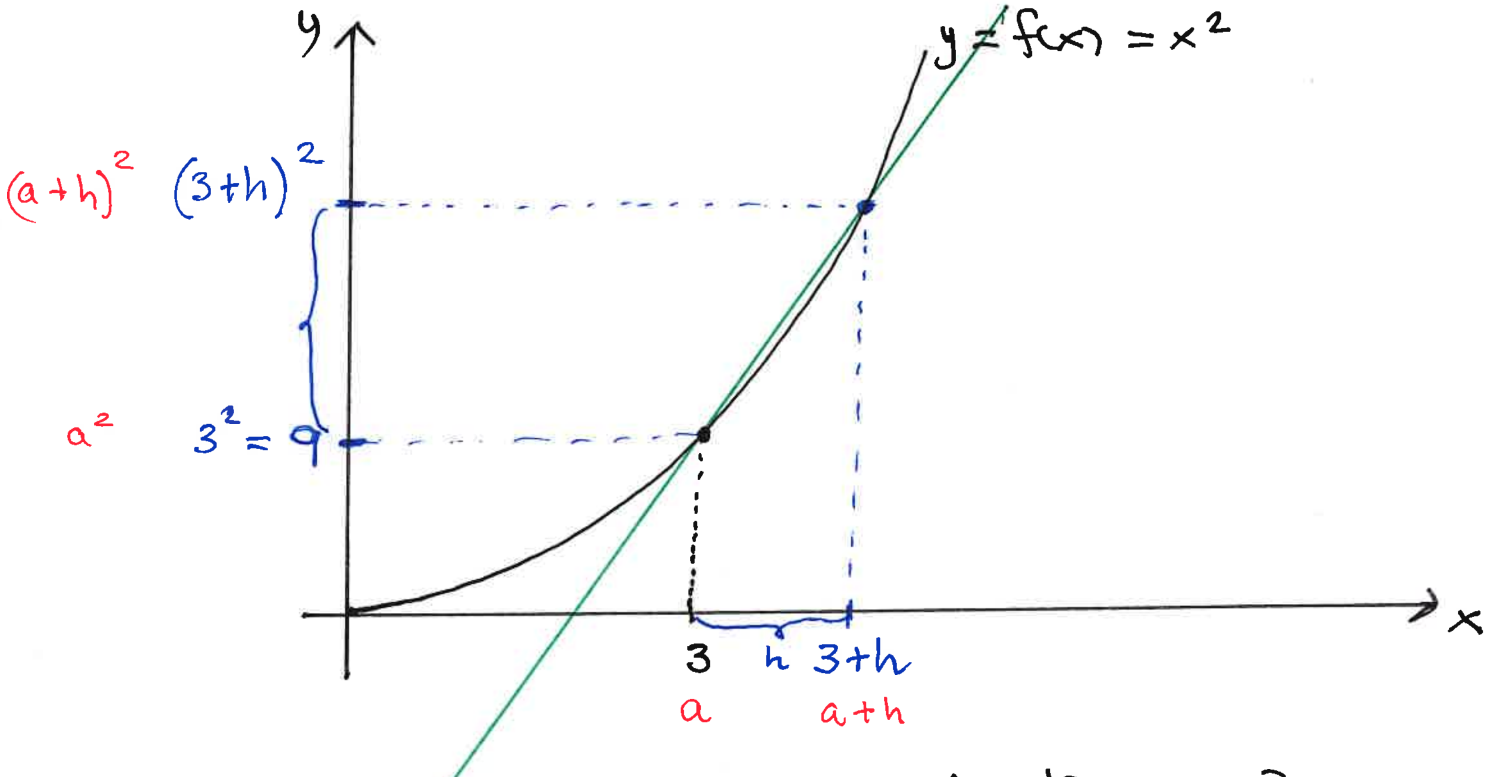
We write $f'(3) = 0.8$

The tangent of the graph of $f(x)$ at the point $(6, 1)$ has slope -0.5

Two important applications
1) To determine where the function increases/decreases and max/min.
2) Approximate complicated functions with linear functions
   – typical in economic models.

How to find the slope of the tangent

<u>Ex</u>: $f(x) = x^2$ and $(3, 9)$. What is the slope of the tangent?



The slope of this ~~secant~~ line is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\overset{(a+h)^2}{(3+h)^2} - \overset{a^2}{3^2}}{h} = \frac{\overset{(a+h)(a+h)-a^2}{(3+h)(3+h)-9}}{h}$$

$$= \frac{\overset{a^2 + 2\cdot a\cdot h + h^2 - a^2}{\cancel{9} + 2\cdot 3\cdot h + h^2 \cancel{-9}}}{h} = \frac{\overset{2ah+h^2}{6h+h^2}}{h} = \frac{\overset{h(2a+h)}{\cancel{h}(6+h)}}{\cancel{h}}$$

$$\overset{2a+h \xrightarrow[h\to 0]{} 2a}{= 6+h \xrightarrow[h\to 0]{} 6}$$ which has to be the slope of the tangent line to $f(x)$ through $(3, 9)$

We write $f'(3) = 6$

also $f'(a) = 2a$

2. The derivative as a function

In the example: If $x = a$, then $f'(a) = 2a$

- but this is a function, and we use x as variable: $f'(x) = 2x$

E.g. The slope of the tangent of $f(x)$ at $(-3, 9)$ is $f'(-3) = 2 \cdot (-3) = -6$

We could do the same thing with similar $f(x) = x^3$. We would (after calculation) get $f'(x) = 3 \cdot x^2$.

3. Rules of differentiation

Power rule: $f(x) = x^n$ gives $f'(x) = n \cdot x^{n-1}$ for all n

Ex $f(x) = x^{10}$, $f'(x) = 10 \cdot x^9$

Ex: $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$, $f'(x) = \frac{1}{3} \cdot x^{\frac{1}{3}-1} = \frac{1}{3} \cdot x^{-\frac{2}{3}}$

$$= \frac{1}{3} \cdot \frac{1}{x^{2/3}} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}}$$

$$= \frac{1}{3 \cdot \sqrt[3]{x^2}}$$

The sum rule If $f(x) = g(x) + h(x)$

then $f'(x) = g'(x) + h'(x)$

Ex $f(x) = x + x^3$, then $f'(x) = 1 + 3x^2$

The constant rule If k is a constant number

and $f(x) = k \cdot g(x)$ then

$f'(x) = k \cdot g'(x)$

Ex $k = 7$, $g(x) = x^2$, then $f(x) = 7x^2$

and $f'(x) = 7 \cdot 2x = 14x$

The product rule If $f(x) = g(x) \cdot h(x)$

then $f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

Ex $f(x) = (5x^3 - 2x + 1)(3x + 7)$

Calculate $f'(x)$ by using the product rule.

$g(x) = 5x^3 - 2x + 1$ $h(x) = 3x + 7$

$g'(x) = 15x^2 - 2$ $h'(x) = 3$

so $f'(x) = (15x^2 - 2) \cdot (3x + 7) + (5x^3 - 2x + 1) \cdot 3$

note the paranthesis!

calculate

$= \underline{\underline{60x^3 + 105x^2 - 12x - 11}}$

The quotient rule

Suppose $f(x) = \dfrac{g(x)}{h(x)}$

Then $$f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

Ex $f(x) = \dfrac{3x+1}{2x+5}$. Then

$g(x) = 3x+1$ and $h(x) = 2x+5$

$g'(x) = 3$ $h'(x) = 2$

note the paranthesis

$$f'(x) = \frac{3 \cdot (2x+5) - (3x+1) \cdot 2}{(2x+5)^2}$$

$$= \frac{3 \cdot 2x + 3 \cdot 5 - (3x \cdot 2 + 1 \cdot 2)}{(2x+5)^2}$$

$$= \frac{6x + 15 - 6x - 2}{(2x+5)^2}$$

$$= \frac{13}{(2x+5)^2}$$

← usually better to not expand denominator !

The chain rule

If $f(x) = g(\overbrace{u(x)}^{\text{the inner function}})$

$\uparrow$ the outer function

then $f'(x) = g'(u) \cdot u'(x)$

where $u = u(x)$

Ex $f(x) = (x^2+2)^{10}$

Put $u = u(x) = x^2+2$ and $g(u) = u^{10}$

$u'(x) = 2x$ $\qquad g'(u) = 10u^9$

Then $f'(x) = 10u^9 \cdot 2x$

$= 10 \cdot (x^2+2)^9 \cdot 2x$

$= \underline{\underline{20x(x^2+2)^9}}$

Two functions

$f(x) = e^x$ and $g(x) = \ln(x)$

$f'(x) = e^x$ $\qquad g'(x) = \dfrac{1}{x}$

Ex $f(x) = e^{3x}$

$u(x) = 3x$ and $g(u) = e^u$

$u'(x) = 3$ $\qquad g'(u) = e^u$

so $f'(x) = 3 \cdot e^{3x}$

Ex $f(x) = \ln(x^2+1)$

then $f'(x) = \dfrac{2x}{x^2+1}$

by the chain rule
with $u = x^2+1$
and $g(u) = \ln(u)$