Suggested solution	EBA 11806 Mathematics for Data Science
Date	December 19th 2024 from 0900 - 1400

Problem 1.

a) We use cofactor expansion along the first row to compute the determinant:

$$\begin{vmatrix} 1 & 2 & 1 \\ t & 0 & 1 \\ 1 & t & t \end{vmatrix} = 1 \cdot (0-t) - 2(t^2-1) + 1(t^2-0) = -t^2 - t + 2$$

We factorize the expression and get $|A| = -(t-1)(t+2)$.

b) When $t = -1$, we have $\det(A) = -(-1)^2 - (-1) + 2 = 2 \neq 0$. Hence, A has an inverse matrix given by

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

where C_{ij} is the cofactor of A in position (i, j). With $t = -1$, the inverse matrix is given by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 1 & -2 & 3 \\ 2 & -2 & 2 \end{pmatrix}^T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 1 & 3 & 2 \end{pmatrix}$$

c) We multiply the equation $A\mathbf{x} = \mathbf{b}$ with A^{-1} from the left. It gives

$$\mathbf{x} = I\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2+3+2 \\ 0+(-6)+(-2) \\ 2+9+2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7 \\ -8 \\ 13 \end{pmatrix}$$

Problem 2.

a) We write down the extended matrix of the system, and use elementary row operations:

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 0 \\ 2 & 3 & 1 & -1 & -1 \\ 4 & 6 & 10 & -10 & 6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 0 \\ 0 & -1 & -3 & 1 & -1 \\ 0 & -2 & 2 & -6 & 6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 0 \\ 0 & -1 & -3 & 1 & -1 \\ 0 & 0 & 8 & -8 & 8 \end{array}\right)$$

The result is an echelon form, and we have marked the pivot positions in blue. Hence, there are infinitely many solutions, with w free, where we write $\mathbf{x} = (x, y, z, w)$ for the unknowns. We find the solutions by back substitution: From the final equation, we get $8z - 8w = 8$, or $z = 1 + w$. The next equation says $-y - 3z + w = -1$, or $y = 1 - 3z + w = 1 - 3(1+w) + w = 1 - 3 - 3w + w = -2 - 2w$. The first equation gives $x + 2y + 2z - w = 0$, or $x = w - 2y - 2z = w - 2(-2-2w) - 2(1+w) = w + 4 + 4w - 2 - 2w = 2 + 3w$. Hence, the solutions of the linear system can be written

$$\mathbf{x} = \begin{pmatrix} 2+3w \\ -2-2w \\ 1+w \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 3 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

where w is a free variable.

b) From (a), we find that

$$(2+3w)\mathbf{v}_1 + (-2-2w)\mathbf{v}_2 + (1+w)\mathbf{v}_3 + w\mathbf{v}_4 = \mathbf{b}$$

for all numbers w. If we let $-2 - 2w = 0$, i.e., $w = -1$, we get

$$(2 + 3 \cdot (-1))\mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + (1 + (-1))\mathbf{v}_3 + (-1)\mathbf{v}_4 = \mathbf{b}.$$

Hence,

$$-\mathbf{v}_1 - \mathbf{v}_4 = \mathbf{b}$$

is a way to write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_3$ and $\mathbf{v}_4$. Alternatively, we can solve the vector equation $x_1\mathbf{v}_1 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{b}$ by writing it as a linear system and using Gaussian elimination.

Problem 3.

a) We use integration by parts, where we differentiate $3x$ and integrate e^x. Hence, we get

$$\int 3xe^x \, \mathrm{d}x = 3xe^x - \int 3e^x \, \mathrm{d}x = 3xe^x - 3e^x + C$$

b) We factorize the denominator of the integrand: $x^2 - x - 6 = (x-3)(x+2)$. Then we use partial fractions to simplify the integrand. From the factorization of the denominator, we can write the integrand in the following way:

$$\frac{3x+1}{x^2-x-6} = \frac{3x+1}{(x-3)(x+2)}$$

We would like to find numbers A and B such that

$$\frac{3x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} \quad \Rightarrow \quad 3x+1 = A(x+2) + B(x-3) = (A+B)x + (2A-3B)$$

This gives $A + B = 3$ and $2A - 3B = 1$ by comparing the coefficients in front of each term on the left and the right hand side. We get $A = 3 - B$ and hence, $2(3-B) - 3B = 1$ or $-5B = -5$, i.e., $B = 1$. This gives $A = 3 - 1 = 2$. Then,

$$\int \frac{3x+1}{x^2-x-6} \, \mathrm{d}x = \int \left(\frac{2}{x-3} + \frac{1}{x+2}\right) \mathrm{d}x = 2\ln|x-3| + \ln|x+2| + C$$

c) We use the substitution $u = \sqrt{x} + 3$, with $\mathrm{d}u = \frac{1}{2\sqrt{x}} \, \mathrm{d}x$. Note that from the substitution, $\sqrt{x} = u - 3$. Hence, we can write $\mathrm{d}x = 2(u-3)\,\mathrm{d}u$. We compute the indefinite integral (the antiderivative)

$$\begin{aligned}\int \frac{5}{\sqrt{x}+3} \, \mathrm{d}x = \int \frac{5}{u} \cdot 2(u-3) \, \mathrm{d}u = \int 10 - \frac{30}{u} \, \mathrm{d}u &= 10u - 30\ln|u| + C \\ &= 10(\sqrt{x}+3) - 30\ln(\sqrt{x}+3) + C\end{aligned}$$

By inserting the integration bounds, we find

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}+1} \, \mathrm{d}x &= 10(\sqrt{4}+3) - 30\ln(\sqrt{4}+3) - \left[10(\sqrt{1}+3) - 30\ln(\sqrt{1}+3)\right] = 50 - 30\ln(5) - [40 - 30\ln(4)] \\ &= 10 - 30\,[\ln(5) - \ln(4)]\end{aligned}$$

d) The total net present value for the rental income during the first 6 years is given by:

$$\int_0^6 I(x)e^{-rx} \, \mathrm{d}x = \int_0^6 300 \cdot e^{0.05x} \cdot e^{-0.08x} \, \mathrm{d}x = 300 \int_0^6 e^{0.05x - 0.08x} \, \mathrm{d}x = 300 \int_0^6 e^{-0.03x} \, \mathrm{d}x$$

By anti-differentiation, we find

$$300 \int_0^6 e^{-0.03x} \, \mathrm{d}x = \frac{300}{-0.03}\left[e^{-0.03x}\right]_{x=0}^{x=6} = \frac{300}{-0.03}\left(e^{-0.03 \cdot 6} - e^0\right) = 10\,000\left(1 - e^{-0.18}\right) = 1647.30$$

Problem 4.

a) The partial derivatives of $f(x\,,y) = (x-3)^2 + 4y^2$ er $f'_x = 2(x-3)$ og $f'_y = 8y$. So the first order conditions $f'_x = 0, f'_y = 0$ are given by

$$\begin{aligned} f'_x = 2(x-3) &= 0 \quad \Rightarrow \quad x = 3, \\ f'_y = 8y &= 0 \quad \Rightarrow \quad y = 0. \end{aligned}$$

Hence, we only have one stationary point for f:

$$(x\,,y) = (3\,,0).$$

The Hessian of f in a general point is given by

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}.$$

The stationary point is $(x\,,y) = (3\,,0)$. In this case, the Hessian is independent of the point, so

$$H(f)(3\,,0) = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}.$$

We see that $\det(H(f)(3\,,0)) = 2 \cdot 8 - 0 \cdot 0 > 0$ and $\operatorname{tr} H(f)(3\,,0) = 2 + 8 > 0$. Hence, $(3\,,0)$ is a local minimum for f.

b) The level curve of $f(x\,,\,y)$ with level $c = 16$ is the set of solutions of the equation $f(x\,,\,y) = (x-3)^2 + 4y^2 = 16$. It can be rewritten as

$$\frac{(x-3)^2}{4^2} + \frac{y^2}{2^2} = 1.$$

We recognise this as the standard equation of an ellipse with centre $(3\,,0)$, horizontal semi-axis $a = 4$ and vertical semi-axis $b = 2$. We can do the corresponding rewriting for $c = 4$:

$$(x-3)^2 + 4y^2 = 4$$

$$\frac{(x-3)^2}{2^2} + \frac{y^2}{1^2} = 1$$

We recognise this as the standard equation of an ellipse with centre $(3\,,0)$, horizontal semi-axis $a = 2$ og vertical semi-axis $b = 1$. This is a smaller ellipse than the one of level $c = 16$.

For $c = 0$ we get $(x-3)^2 + 4y^2 = 0$. Because $(x-3)^2 \geq 0$ and $4y^2 \geq 0$ for all $(x\,,\,y)$, this means that $x - 3 = 0$, i.e., $x = 3$ and $y = 0$. Hence the level curve for $c = 0$ is only the point $(3\,,0)$. This is the centre of the ellipses above. In Figure 1 the level curves are graphed in the same coordinate system. The level curve of each level $c > 0$ of the function f is an ellipse in the xy-plane with

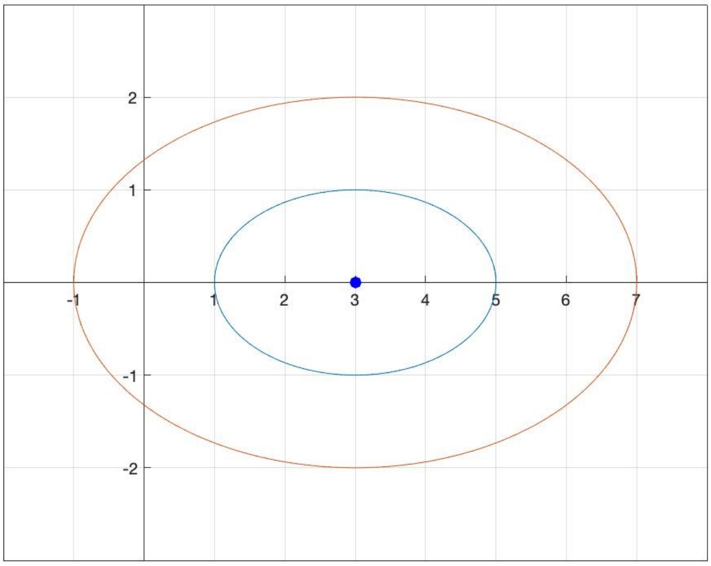

FIGURE 1. Level curves for f.

the same centre $(x\,,\,y) = (3\,,\,0)$. The larger the ellipse, the larger the function value. Since the function is defined for all $(x\,,\,y)$, this means that there is no maximum for $f(x)$: We can always make larger and larger level curve ellipses which correspond to larger and larger function values.

However, we see that the equation

$$(x-3)^2 + 4y^2 = c$$

has no solutions if $c < 0$ because the left hand side is a sum of two square numbers and hence, greater or equal to 0. Consequently, we get that f has a global minimum point in $(x\,,\,y) = (3\,,0)$ with level $c = 0 = f(3\,,0)$. Note that this minimum point is the same we found in (a).

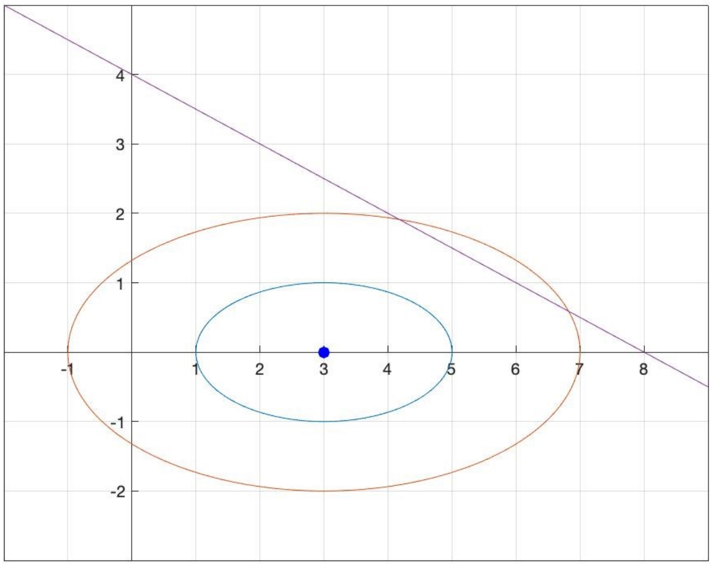

FIGURE 2. Level curves for f with the constraint.

c) The constraint $0.5x+y=4$ can be written as $y=4-0.5x$. In Figure 2, the graph of this function is drawn in the same coordinate system as the level curves. The constraint means that the domain of definition of $f(x\,,\,y)$ now are the points on the graph of $y=4-0.5x$. From the figure, we see that the optimization problem with the constraint still has no maximum: We can get to an arbitrarily large level curve ellipse by following the straight line. However, the optimization problem with constraint has a minimum. We can find this by drawing a level curve ellipse which is tangent to the straight line. The point where the tangent meets the level curve will be the minimum point. The minimal value (which is the level at the tangent point) will be larger than 4 and less than 16.

Problem 5.

a) We use the Lagrange multiplier method with $\mathcal{L}=x^2y^2+22x^2+2y^2-\lambda(x^2+y^2-52)$ in order to find candidate points. The Lagrange conditions are

$$\mathcal{L}'_x = 2xy^2+44x-2\lambda x=0$$
$$\mathcal{L}'_y = 2x^2y+4y-2\lambda y=0$$
$$x^2+y^2=52$$

We have three cases:

- If $x\,,\,y\neq 0$, we can divide wrt. $2x$ and $2y$ respectively, and hence, solve the two first equations for λ. In this case, we get:
$$\lambda = y^2+22 = x^2+2$$
Hence, $x^2=y^2+20$. We insert this into the constraint, and get $x^2+y^2=y^2+20+y^2=2y^2+20=52$, so $y^2=16 \implies y=\pm 4$. This means that $x^2=y^2+20=16+20=36$, so $x=\pm 6$. We compute the corresponding λ from f.ex. $\lambda=y^2+22=16+22=38$. This gives four candidate points:
$$(6\,,4\,;\,38)\,,(6\,,\,-4\,;\,38)\,,(-6\,,4\,;\,38)\,,(-6\,,\,-4\,;\,38).$$
The corresponding function values are
$$f(6\,,4)=f(6\,,-4)=f(-6\,,4)=f(-6\,,-4)=36\cdot 16+22\cdot 36+2\cdot 16=1400.$$
- If $x=0$, the first equation automatically holds. From the second equation, we get $4y-2\lambda y=0$, i.e., $y(2-\lambda)=0$. In order for this to hold, we need $y=0$ or $\lambda=2$ (or both). If $y=0$, the constraint cannot hold, since this gives $x^2+y^2=0^2+0^2\neq 52$. However, $\lambda=2$ is possible. In

this case, we find y by solving $0^2 + y^2 = 52$, which gives $y = \pm\sqrt{52}$. The candidate point is $(x\,, y\,;\,\lambda) = (0\,,\,\pm\sqrt{52}\,;\,2)$. The corresponding function value is $f(0\,,\,\pm\sqrt{52}) = 2 \cdot 52 = 104$.

- If $y = 0$ the second equation is automatically true. A similar argument as above, but for the first equation, implies that the only possibility is $\lambda = 22$ with corresponding $x = \pm\sqrt{52}$. The candidate point is $(x\,,\,y\,;\,\lambda) = (\pm\sqrt{52}\,, 0\,;\,22)$. The corresponding function value is $f(\pm\sqrt{52}\,, 0) = 1144$.

b) The Extreme Value Theorem says that any continuous function defined on a closed and bounded set has a (global) maximum and a (global) minimum.

The Extreme Value Theorem can be applied to the Lagrange problem, because $f(x, y)$ is a continuous function, and the set of admissible points is closed and bounded: It is closed because it is defined by an equality (i.e., contains its boundary). The set is bounded because $x^2 + y^2 = 52$ is a circle with center in the origin and radius $\sqrt{52}$.

c) A point has degenerated constraint if $g'_x = g'_y = 0$, where $g(x,y) = x^2 + y^2$. This gives

$$g'_x = 2x = 0,\; g'_y = 2y = 0 \quad \Rightarrow \quad x = y = 0$$

But the point $(x\,, y) = (0\,, 0)$ is not admissible, since it does not satisfy the constraint; $g(0\,, 0) = 0 \neq 52$. We conclude that there are no feasible points with degenerated constraint for this problem.

From the Extreme Value Theorem it follows that the Lagrange problem has a maximum and a minimum. These have to be among the candidate points we have found, since there are no admissible points with degenerated constraint. By finding the largest and the smallest function value among the candidate points from (a), it follows that the maximum value is $f_{\max} = 1400$ in the points

$$(x\,, y) = (6\,, 4), (6\,, -4), (-6\,, 4), (-6\,, -4)$$

med $\lambda = 38$. Similarily, the minimum value is $f_{\min} = 104$ in the points $(x\,, y) = (0, \pm\sqrt{52})$ with $\lambda = 22$.