Exam	EBA 1180 Mathematics for Data Science
Date	May 27th 2024, from 0900 to 1400

The exam consists of 15 subproblems with equal weight. All answers must be justified.

Problem 1.
Consider the matrix A and the vector $\mathbf{b}$ given by

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & a & 0 \\ a & 1 & a \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(a) Compute the determinant of A, and factorize the expression.
(b) Find A^{-1} when $a = 2$.
(c) Solve the equation $A^{-1} \cdot \mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ when $a = 2$.

Problem 2.
Consider the matrix A and the vector $\mathbf{b}$ given by

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 1 & 1 & -1 & 0 \\ 3 & 4 & 8 & -9 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -2 \\ 12 \end{pmatrix}$$

and let the column vectors of A be denoted by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

(a) Use Gaussian elimination to solve the linear system $A\mathbf{x} = \mathbf{b}$.
(b) Write $\mathbf{b}$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$ if possible.

Problem 3.
Compute the following integrals:

a) $\displaystyle\int xe^{-x}\, dx$ b) $\displaystyle\int \frac{3x+4}{x^2+x-6}\, dx$ c) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}+1}\, dx$

Let $I(x) = 200 \cdot 1.04^x$ be the cash flow after x years when renting out a property. We view this as a continuous cash flow, and we use continuous discounting with discount rate $r = 6\%$ when computing the net present value.

d) Find the total present value of the rental income during the first five years.

Problem 4.
Consider the function f given by $f(x,y) = 9(x-3)^2 + 4y^2$.

(a) Find the stationary points of f. Compute the Hessian of f and use this to classify any stationary points you found.
(b) Draw level curves for $f(x,y)$ at the levels $c = 0, 1, 4$ and 9. Use this figure to explain whether f has maximum- and/or minimum values.
(c) Assume that we instead consider the optimization problem $\max / \min f(x,y)$ with the constraint $x + y = 4$. Draw the graph of this constraint into the same coordinate system as the level curves. Use the figure to estimate the maximum- and/or minimum values for the Lagrange problem.

Problem 5.
Consider the Lagrange problem $\max / \min f(x,y) = x^2y^2 + 2x^2 + y^2$ when $x^2 + y^2 = 33$.

(a) Find all points $(x,y;\lambda)$ that satisfy the Lagrange conditions.
(b) State the Extreme Value Theorem (EVT). Can the Extreme Value Theorem be applied to this Lagrange problem?
(c) Solve the Lagrange problem. Remember to give a complete justification for your answer.

Formula Sheet

Financial mathematics

Geometric series.
A finite geometric series has sum

$$S_n = a_1 \cdot \frac{1-k^n}{1-k} = a_1 \cdot \frac{k^n - 1}{k-1}$$

and an infinite geometric series has sum

$$S = a_1 \cdot \frac{1}{1-k} \quad \text{when } |k| < 1$$

Present values.
The present value K_0 of a payment K is given by

$$K_0 = \frac{K_n}{(1+r)^n} \quad \text{and} \quad K_0 = \frac{K_n}{e^{rn}}$$

using discrete and continuous compounding.

Integration

Integration techniques.

a) Integration by parts:

$$\int u'v \, \mathrm{d}x = uv - \int uv' \, \mathrm{d}x$$

b) Substitution:

$$\int f(u)u' \, \mathrm{d}x = \int f(u) \, \mathrm{d}u$$

c) Partial fractions:

$$\int \frac{px+q}{(x-a)(x-b)} \, \mathrm{d}x = \int \left(\frac{A}{x-a} + \frac{B}{x-b} \right) \mathrm{d}x$$

Area.
The area of the region bounded by $a \le x \le b$ and $f(x) \le y \le g(x)$ is given by

$$A = \int_a^b (g(x) - f(x)) \, \mathrm{d}x$$

Linear algebra

Cramer's rule.
A linear system $A\mathbf{x} = \mathbf{b}$ where $|A| \neq 0$ has a unique solution given by

$$x_1 = \frac{|A_1(\mathbf{b})|}{|A|} \quad x_2 = \frac{|A_2(\mathbf{b})|}{|A|} \quad \ldots \quad x_n = \frac{|A_n(\mathbf{b})|}{|A|}$$

where $A_i(\mathbf{b})$ is the matrix obtained by replacing column i of A by $\mathbf{b}$.

Functions of two variables

Second derivative test.
A stationary point (x^*, y^*) of the function $f(x,y)$ is a

a) local minimum if $A > 0$ and $AC - B^2 > 0$
b) local maximum if $A < 0$ and $AC - B^2 > 0$
c) saddle point if $AC - B^2 < 0$

when $H(f)(x^*, y^*) = \left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right)$.

Level curves.
The slope $y' = \mathrm{d}y/\mathrm{d}x$ of the tangent line to the level curve $f(x,y) = c$ is given by

$$y' = -\frac{f'_x}{f'_y}$$

Method of Lagrange multipliers.
The Lagrange conditions for the problem

$$\max / \min f(x,y) \text{ when } g(x,y) = a$$

is given by

$$\mathcal{L}'_x = 0, \ \mathcal{L}'_y = 0, \ g(x,y) = a$$

An admissible point has degenerated constraint if

$$g'_x = 0, \ g'_y = 0$$