Suggested solution	EBA 1180 Mathematics for Data Science
Date	May 27th 2024 from 0900 - 1400

Problem 1.

(a) We use cofactor expansion along the first row to compute the determinant:

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & a & 0 \\ a & 1 & a \end{vmatrix} = 1(a^2 - 0) - 1(a - 0) + 2(1 - a^2) == -a^2 - a + 2$$

By factorizing (e.g., via the abc-formula), we find that the determinant can be written as $|A| = -(a-1)(a+2)$.

(b) When $a = 2$, we have $\det(A) = -2^2 - 2 + 2 = -4 \neq 0$. Hence, A has an inverse matrix given by

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

where C_{ij} is the cofactor of A in position (i,j). With $a = 2$, the inverse matrix is given by

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{-4} \begin{pmatrix} 4 & -2 & -3 \\ 0 & -2 & 1 \\ -4 & 2 & 1 \end{pmatrix}^T = \frac{1}{4} \begin{pmatrix} -4 & 0 & 4 \\ 2 & 2 & -2 \\ 3 & -1 & -1 \end{pmatrix}$$

(c) Note that,

$$\begin{aligned} A^{-1}\mathbf{x} &= \mathbf{b} \\ AA^{-1}\mathbf{x} &= A\mathbf{b} \\ I\mathbf{x} &= A\mathbf{b} \\ \mathbf{x} &= A\mathbf{b} \end{aligned}$$

Hence, we can solve the equation by calculating $A\mathbf{b}$:

$$\mathbf{x} = A\mathbf{b} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+2+6 \\ 1+4+0 \\ 2+2+6 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 10 \end{pmatrix}$$

Problem 2.

(a) We formulate the extended matrix of the system, and use elementary row operations:

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 0 \\ 1 & 1 & -1 & 0 & -2 \\ 3 & 4 & 8 & -9 & 12 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 0 \\ 0 & -1 & -3 & 1 & -2 \\ 0 & -2 & 2 & -6 & 12 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 0 \\ 0 & -1 & -3 & 1 & -2 \\ 0 & 0 & 8 & -8 & 16 \end{array}\right)$$

The result is an echelon form, and we have marked the pivot positions in blue. Hence, there are infinitely many solutions, with w free, where we write $\mathbf{x} = (x,y,z,w)$ for the unknowns. We find the solutions by back substitution: From the final equation, we get $8z - 8w = 16$, or $z = 2 + w$. The next equation says $-y - 3z + w = -2$, or $y = 2 - 3z + w = 2 - 3(2 + w) + w = 2 - 6 - 3w + w = -4 - 2w$. The first equation gives $x + 2y + 2z - w = 0$, or $x = w - 2y - 2z = w - 2(-4 - 2w) - 2(2 + w) = w + 8 + 4w - 4 - 2w = 4 + 3w$. Hence, the solutions of the linear system can be written

$$\mathbf{x} = \begin{pmatrix} 4 + 3w \\ -4 - 2w \\ 2 + w \\ w \end{pmatrix} = w \begin{pmatrix} 3 \\ -2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \\ 2 \\ 0 \end{pmatrix}$$

where w is a free variable.

(b) From (a), we find that

$$(4+3w)\mathbf{v}_1 + (-4-2w)\mathbf{v}_2 + (2+w)\mathbf{v}_3 + w\mathbf{v}_4 = \mathbf{b}$$

for all numbers w. If we let $-4-2w=0$, i.e., $w=-2$, we get

$$(4+3\cdot(-2))\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + (2+(-2))\mathbf{v}_3 + (-2)\mathbf{v}_4 = \mathbf{b}.$$

Hence,

$$-2\mathbf{v}_1 - 2\mathbf{v}_4 = \mathbf{b}$$

is a way to write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_3$ and $\mathbf{v}_4$. Alternatively, we can solve the vector equation $x_1\mathbf{v}_1 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4$ by writing it as a linear system and using Gaussian elimination.

Problem 3.

(a) We use integration by parts, where we integrate e^{-x} and differentiate x. Hence, we get

$$\int xe^{-x}\,\mathrm{d}x = -e^{-x}x - \int(-e^{-x})\cdot 1\,\mathrm{d}x = -xe^{-x} + \int e^{-x}\,\mathrm{d}x = -xe^{-x} - e^{-x} + C$$

(b) We factorize the denominator of the integrand, for example via the abc-formula, and get

$$x^2+x-6 = (x-2)(x+3).$$

Then, we use partial fractions to simplify the integrand. From the factorization of the denominator, we can write the integrand in the following way:

$$\frac{3x+4}{x^2+x-6} = \frac{3x+4}{(x-2)(x+3)}$$

We would like to find numbers A and B such that

$$\frac{3x+4}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3} \quad \Rightarrow \quad 3x+4 = A(x+3) + B(x-2) = (A+B)x + (3A-2B)$$

This gives $A+B=3$ and $3A-2B=4$ by comparing the coefficients in front of each term on the left and the right hand side. We get $A=3-B$ and hence $3(3-B)-2B=4$ or $-5B=-5$, i.e., $B=1$. This gives $A=3-1=2$. Then,

$$\int \frac{3x+4}{x^2+x-6}\,\mathrm{d}x = \int\left(\frac{2}{x-2} + \frac{1}{x+3}\right)\mathrm{d}x = 2\ln|x-2| + \ln|x+3| + C$$

(c) We use the substitution $u=\sqrt{x}+1$, with $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$. Note that from the substitution, $\sqrt{x}=u-1$. Hence, we can write $\mathrm{d}x = 2(u-1)\,\mathrm{d}u$. We compute the new integration bounds after the substitution: $x=0$ which gives $u=\sqrt{0}+1=1$ and $x=1$ which gives $u=\sqrt{1}+1=2$. From this, we get:

$$\int_0^1 \frac{1}{\sqrt{x}+1}\,\mathrm{d}x = \int_1^2 \frac{1}{u}2(u-1)\,\mathrm{d}u = \int_1^2\left(2-\frac{2}{u}\right)\mathrm{d}u = \left[2u - 2\ln|u|\right]_{u=1}^2$$

By inserting the new integration bounds, we find

$$\int_0^1 \frac{1}{\sqrt{x}+1}\,\mathrm{d}x = (4-2\ln 2) - (2-2\ln 1) = 4 - 2\ln 2 - 2 + 0 = 2 - 2\ln 2$$

(d) The total net present value for the rental income during the first 5 years is given by:

$$\int_0^5 I(x)e^{-rx}\,\mathrm{d}x = \int_0^5 200\cdot 1{,}04^x e^{-0{,}06x}\,\mathrm{d}x = 200\int_0^5 (e^{\ln 1{,}04})^x e^{-0{,}06x}\,\mathrm{d}x = 200\int_0^5 e^{(\ln 1{,}04-0{,}06)x}\,\mathrm{d}x$$

By anti-differentiation, we find

$$\int_0^5 I(x)e^{-rx}\,\mathrm{d}x = \frac{200}{\ln 1{,}04 - 0{,}06}\left[e^{(\ln 1{,}04-0{,}06)x}\right]_{x=0}^5 = \frac{200}{\ln 1{,}04-0{,}06}\left(e^{5(\ln 1{,}04-0{,}06)} - 1\right) \approx 950$$

Problem 4.

(a) The partial derivatives of $f(x,y) = 9(x-3)^2 + 4y^2$ is $f'_x = 18(x-3)$ and $f'_y = 8y$. Hence, the first order conditions $f'_x = 0$, $f'_y = 0$ are given by

$$\begin{aligned} f'_x &= 18(x-3) = 0 \quad \Rightarrow \quad x = 3, \\ f'_y &= 8y = 0 \quad \Rightarrow \quad y = 0. \end{aligned}$$

Hence, we only have one stationary point for f:

$$(x,y) = (3,0).$$

The Hessian of f in a general point is given by

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 18 & 0 \\ 0 & 8 \end{pmatrix}.$$

The stationary point is $(x,y) = (3,0)$. In this case, the Hessian is independent of the point, so

$$H(f)(3,0) = \begin{pmatrix} 18 & 0 \\ 0 & 8 \end{pmatrix}.$$

We see that $\det(H(f)(3,0)) = 18 \cdot 8 - 0 \cdot 0 > 0$ and $\operatorname{tr} H(f)(3,0) = 18 + 8 > 0$. Hence, (3,0) is a local minimum for f.

(b) $f(x,y) = 9(x-3)^2 + 4y^2 = 1$ can be rewritten as

$$\frac{(x-3)^2}{(\frac{1}{3})^2} + \frac{y^2}{(\frac{1}{2})^2} = 1.$$

We recognize this as the standard equation of an ellipse with center in the point (3,0) with horizontal half axis $\frac{1}{3}$ and vertical half axis $\frac{1}{2}$.

We can rewrite in a similar way for $c = 4$ and $c = 9$. For $c = 4$, we get:

$$9(x-3)^2 + 4y^2 = 4$$

$$\frac{(x-3)^2}{(\frac{1}{3})^2} + \frac{y^2}{(\frac{1}{2})^2} = 4$$

By dividing by 4 on both sides of the equation, we get $\frac{(x-3)^2}{(\frac{2}{3})^2} + \frac{y^2}{(\frac{2}{2})^2} = 1$. Hence,

$$\frac{(x-3)^2}{(\frac{2}{3})^2} + \frac{y^2}{1^2} = 1.$$

We recognize this as the standard equation of an ellipse with center in (3,0), horizontal half axis $\frac{2}{3}$ and vertical half axis 1. This is a larger ellipse than the one corresponding to level $c = 1$. Similar calculations for $c = 9$ gives $9(x-3)^2 + 4y^2 = 9$, which can be rewritten as $\frac{(x-3)^2}{(\frac{3}{3})^2} + \frac{y^2}{(\frac{3}{2})^2} = 1$. Hence,

$$\frac{(x-3)^2}{1^2} + \frac{y^2}{(\frac{3}{2})^2} = 1.$$

This is the standard equation of an ellipse with center in (3,0), horizontal half axis 1 and vertical half axis $\frac{3}{2}$. Again, this is a larger ellipse than the one corresponding to level $c = 4$.

For $c = 0$, we get $9(x-3)^2 + 4y^2 = 0$. Since $9(x-3)^2 \geq 0$ and $4y^2 \geq 0$ for all (x,y), this implies that $x - 3 = 0$, i.e., $x = 3$ and $y = 0$. Hence, the level curve for $c = 0$ is just the point (3,0). This is the center of the ellipses derived above. In Figure 1, the level curves are graphed in the same coordinate system.

From the level curves, we see that each level of the function f corresponds to an ellipse in the xy-plane. The larger the ellise, the larger the function value. Since the function is defined for all (x,y), this means that there is no maximum: We can always make larger and larger level curve ellipses which correspond to larger and larger function values. However, we see

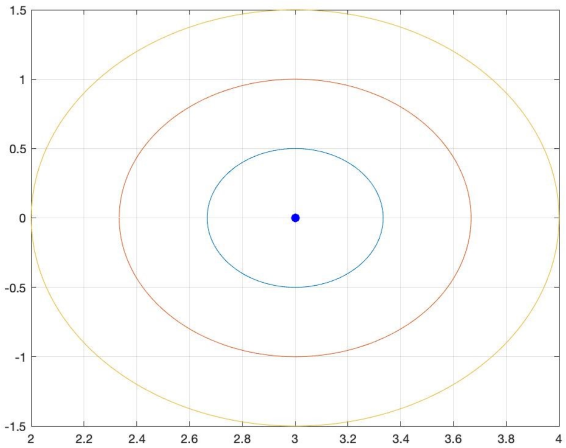

FIGURE 1. Level curves for f.

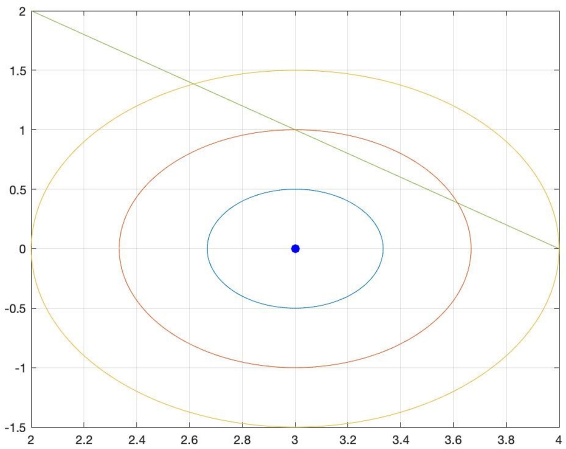

FIGURE 2. Level curves for f with the constraint.

that f has a minimum point in $(x,y) = (3,0)$. From (b), we know that this corresponds to level $c = 0 = f(3,0)$. Note that this minimum point is the same as the one we found in (a).

(c) The constraint $x + y = 4$ can be written as $y = 4 - x$. In Figure 2, the graph of this function is drawn in the same coordinate system as the level curves.

The constraint means the we can only choose among points on the graph of $y = 4 - x$. From the figure, we see that the optimization problem with the constraint still has no maximum: We can get to an arbitrarily large level curve ellipse by following the straight line. However, the optimization problem with constraint has a minimum. We can find this by drawing a level curve ellipse which is tangent to the straight line. The point where the tangent meets the level curve will be the minimum point. From the figure, we see that this point is between the level curve corresponding to level $c = 1$ and level $c = 4$, so $f_{\min}$ is between 1 and 4. Note that the minimum value for the constrained optimization problem will be greater than the one found in the corresponding unbounded problem.

Problem 5.

(a) We use the Lagrange multiplier method with $\mathcal{L} = x^2y^2 + 2x^2 + y^2 - \lambda(x^2 + y^2 - 33)$ in order to find candidate points. The Lagrange conditions are

$$\begin{aligned} \mathcal{L}'_x &= 2xy^2 + 4x - 2\lambda x = 0 \\ \mathcal{L}'_y &= 2x^2y + 2y - 2\lambda y = 0 \\ x^2 + y^2 &= 33 \end{aligned}$$

We have three cases:

- If $x,y \neq 0$, we can divide wrt. $2x$ and $2y$ respectively, and hence solve the two first equations for λ. In this case, we get:
$$\lambda = y^2 + 2 = x^2 + 1$$
Hence, $x^2 = y^2 + 1$. We insert this into the constraint, and get $x^2 + y^2 = y^2 + 1 + y^2 = 2y^2 + 1 = 33$, so $y^2 = 16 \implies y = \pm 4$. This means that $x^2 = y^2 + 1 = 16 + 1 = 17$, so $x = \pm\sqrt{17}$. We compute the corresponding λ from f.ex. $\lambda = y^2 + 2 = 16 + 2 = 18$. This gives four candidate points:
$$(\sqrt{17},4;18),(\sqrt{17},-4;18),(-\sqrt{17},4;18),(-\sqrt{17},-4;18).$$
The corresponding function values are
$$f(\sqrt{17},4) = f(\sqrt{17},-4) = f(-\sqrt{17},4) = f(-\sqrt{17},-4) = 16 \cdot 17 + 2 \cdot 17 + 16 = 322$$
.
- If $x = 0$, the first equation automatically holds. From the second equation, we get $2y - 2\lambda y = 0$, i.e., $y(1-\lambda) = 0$. In order for this to hold, we need $y = 0$ or $\lambda = 1$ (or both). If $y = 0$, the constraint cannot hold, since this gives $x^2 + y^2 = 0^2 + 0^2 \neq 33$. However, $\lambda = 1$ is possible. In this case, we find y by solving $0^2 + y^2 = 33$, which gives $y = \pm\sqrt{33}$. The candidate point is $(x,y;\lambda) = (0,\pm\sqrt{33};1)$. The corresponding function value is $f(0,\pm\sqrt{33}) = 33$.
- If $y = 0$, a similar argument as above implies that the only possibility is $\lambda = 2$ with corresponding $x = \pm\sqrt{33}$. The candidate point is $(x,y;\lambda) = (\pm\sqrt{33},0;2)$. The corresponding function value is $f(\pm\sqrt{33},0) = 66$.

(b) The Extreme Value Theorem says that any continuous function defined on a closed and bounded set has a (global) maximum and a (global) minimum.

The Extreme Value Theorem can be applied to the Lagrange problem, because $f(x,y)$ is a continuous function, and the set of admissible points is closed and bounded: It is closed because it is defined by an equality (i.e., contains its boundary). The set is bounded because $x^2 + y^2 = 33$ is a circle with center in the origin and radius $\sqrt{33}$.

(c) A point has degenerated constraint if $g'_x = g'_y = 0$, where $g(x,y) = x^2 + y^2$. This gives
$$g'_x = 2x = 0, \; g'_y = 2y = 0 \quad \Rightarrow \quad x = y = 0$$
But the point $(x,y) = (0,0)$ is not admissible, since it does not satisfy the constraint; $g(0,0) = 0 \neq 33$. We conclude that there are no feasible points with degenerated constraint for this problem.

From the Extreme Value Theorem it follows that the Lagrange problem has a maximum and a minimum. These have to be among the candidate points we have found, since there are no admissible points with degenerated constraint. By finding the largest and the smallest function value among the candidate points from (a), it follows that the maximum value is $f_{\max} = 322$ in the points
$$(x,y) = (\sqrt{17},4),(\sqrt{17},-4),(-\sqrt{17},4),(-\sqrt{17},-4)$$
med $\lambda = 18$. Tilsvarende er minimumsverdien $f_{\min} = 33$ i punktene $(x,y) = (0,\pm\sqrt{33})$ med $\lambda = 1$.