School exam (3h) EBA11805 - Mathematics for Data Science

12 Des. 2023

SOLUTIONS

Problem 1

This is a geometric series with first term $a_1 = 7\,000 \cdot 1.004^{20}$, multiplication factor $k = 1.004$ and $n = 91 - 19 = 72$ terms. The formula for the sum gives

$$7\,000 \cdot 1.004^{20} \cdot \frac{1.004^{72} - 1}{0.004} = \underline{\underline{631\,168.36}}$$

Problem 2

i) We use the product rule: $f'(x) = (12x)' \cdot e^x + 12x \cdot (e^x)' = 12e^x + 12xe^x = \underline{\underline{12(1+x)e^x}}$

ii) We use the fraction rule: $f'(x) = \frac{(36-4x)'(x-7)-(36-4x)(x-7)'}{(x-7)^2} = \frac{-4(x-7)-(36-4x)\cdot 1}{(x-7)^2} = \underline{\underline{\frac{-8}{(x-7)^2}}}$

iii) We have $\ln(x^{50}) = 50 \cdot \ln(x)$ hence $f'(x) = \underline{\underline{\frac{50}{x}}}$

Problem 3

i) We use polynomial division and get $f(x) = -4 + \frac{8}{x-7}$. This is the standard form for a hyperbola function with vertical asymptote the line $\underline{\underline{x = 7}}$ og horizontal asymptote the line $\underline{\underline{y = -4}}$.

ii) The symmetry point is $(7, -4)$. We calculate some values

x	-1	15	3	11	5	9
$f(x)$	-5	-3	-6	-2	-8	0

and draw the graph and the asymptotes with all the data we have.

Figure 1: The graph of $f(x)$

Problem 4

i) We put both sides of the inequality into e^x. Because e^x is a strictly increasing function we get an equivalent inequality: $e^{\ln(x+5)} \geqslant e^3$. Moreover, because e^x and $\ln(x)$ are inverse functions this gives $x+5 \geqslant e^3$, that is $\underline{\underline{x \geqslant e^3 - 5}}$.

ii) If we deposit 1 krone into an account with 24% interest and annual compounding the balance after 100 000 years will be $1.24^{100\,000}$. The amount $1.02^{1\,200\,000}$ is the balance after 100 000 years with monthly compounding, and is thus the larger number.
Alternative: $1.02^{1\,200\,000} = (1.02^{12})^{100\,000} = 1.26\ldots^{100\,000} > 1.24^{100\,000}$.

Problem 5

i) Let r denote the internal rate of return. Then the present value of the cash flow equals 0. This gives the equation

$$\underline{\underline{-20 + \frac{10}{(1+r)^5} + \frac{25}{(1+r)^6} = 0}}$$

ii) If we insert $r = 10\%$ into the left hand side it equals 0.321. If r increases the powers in the denominators also increase and the value of the positive fractions decreases (the present value of future payments decreases if the interest increases). Hence the internal rate of return is (a little) greater than 10%.

Problem 6

i)

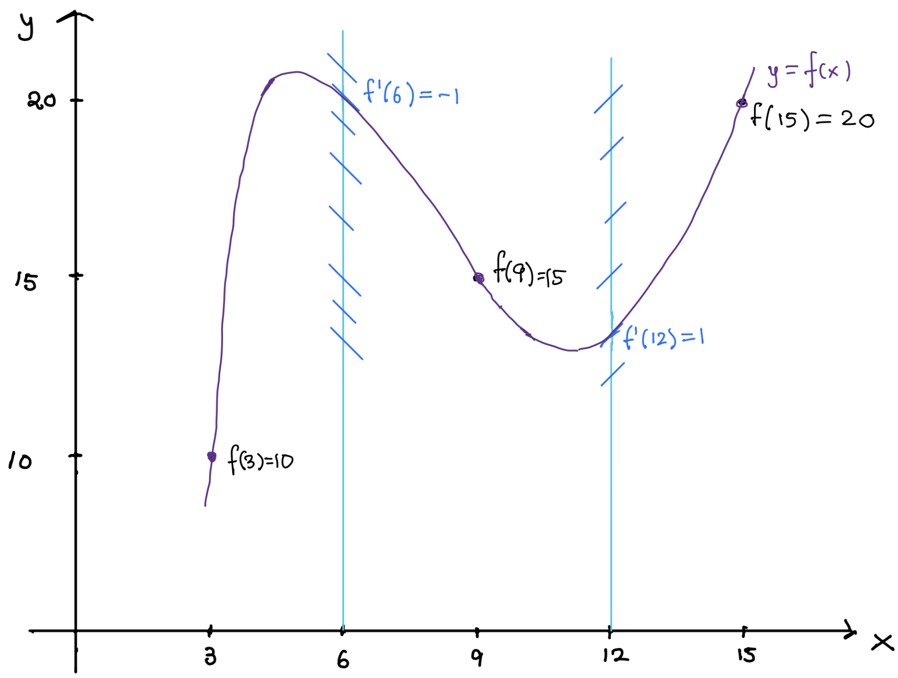

Figure 2: One possible $f(x)$

ii)

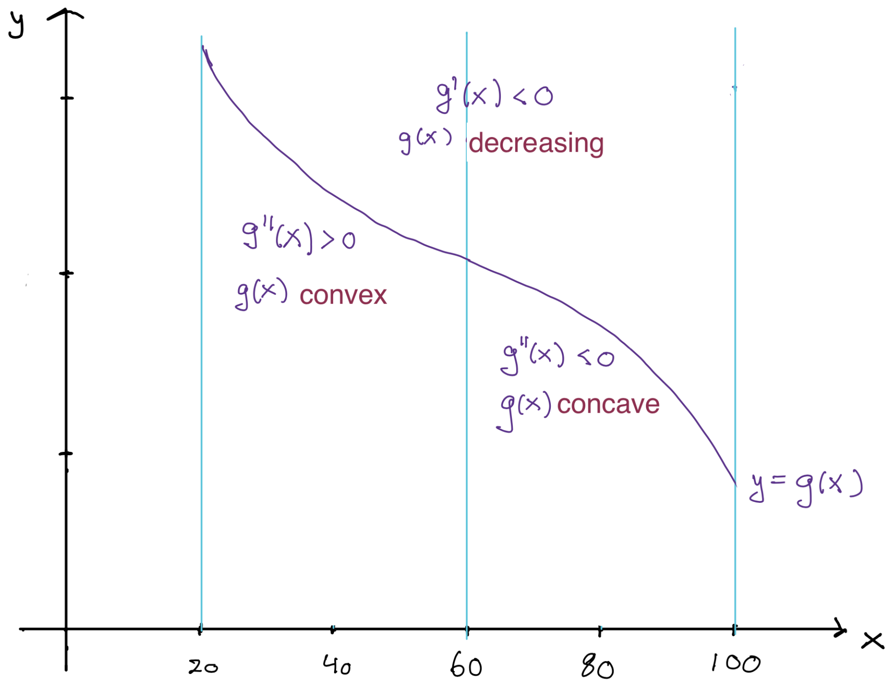

Figure 3: One possible $g(x)$

Problem 7

i) False. Because $f'(x)$ is at least 0.6 in the interval $[9,11]$, $f(x)$ is strictly increasing and $f(9) < f(11)$.

ii) True. Stationary points for $f(x)$ are points where $f'(x)$ equals 0. In the interval $[2,8]$ this happens for $x \approx 2.4$, $x \approx 4.2$ and $x \approx 7$.

iii) True. The inflection points for $f(x)$ is where $f''(x)$ changes sign. It is the same as where $f'(x)$ has (local) maximum- or minimum points. In the interval $[2,11]$, $f'(x)$ has two maximum points ($x \approx 3$ and $x \approx 9.4$) and one minimum point ($x \approx 5.3$).

Problem 8

i) Differentiate (uses the chain rule on the second term with $u(x) = rx$ and $g(u) = 100e^u$) and get the marginal cost function $C'(x) = 3 + 100re^{rx}$.

ii) Because the cost function is strictly convex ($C''(x) = 100r^2e^{rx} > 0$) we get cost optimum as the solution of the equation $C'(x) = A(x)$ where $A(x) = (3x + 100e^{rx})/x$ is average unit cost – this is a result. Hence

$$3 + 100re^{rx} = \frac{3x + 100e^{rx}}{x} \quad \text{that is} \quad 3 + 100re^{rx} = 3 + \frac{100e^{rx}}{x} \quad \text{that is} \quad 100re^{rx} \cdot x = 100e^{rx}$$

that is $rx = 1$ hence $x = r^{-1}$.

Minimal average unit cost is then $A(r^{-1}) = C'(r^{-1}) = 3 + 100re^{r \cdot r^{-1}} = 3 + 100er$

Problem 9

i) We have $D'(p) = 14(p-60)$ hence

$$\varepsilon(p) = \frac{D'(p)\cdot p}{D(p)} = \frac{14(p-60)p}{7(p-60)^2} = \underline{\underline{\frac{2p}{p-60}}}$$

ii) We have $\varepsilon(24) = \frac{2\cdot 24}{24-60} = -1.33$ which is less than -1. The demand is thus elastic w.r.t. the price for $p = 24$ and hence the revenue is going down if the price is increased a little.

Problem 10

i) We have $f(x) = x^{0.5}$ and by the power rule $f'(x) = 0.5x^{-0,5}$ and $f''(x) = -0,25x^{-1.5}$. With $x = 9$ we get $f(9) = \sqrt{9} = 3$, $f'(9) = 0.5\cdot 9^{-0.5} = \frac{1}{2\cdot\sqrt{9}} = \frac{1}{6}$ and $f''(9) = -0.25\cdot 9^{-1.5} = \frac{-1}{4\cdot 9\sqrt{9}} = \frac{-1}{108}$. Inserted into the formula for the Taylor polynomial $P_2(x)$ of degree 2 about $x = 9$ we get

$$P_2(x) = f(9) + f'(9)(x-9) + \frac{1}{2}f''(9)(x-9)^2 = 3 + \frac{1}{6}(x-9) + \frac{1}{2}\cdot\frac{-1}{108}(x-9)^2$$
$$= \underline{\underline{3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2}}$$

ii) We insert $x = 10$ into $P_2(x)$ to determine an approximate value for $\sqrt{10}$.

$$\sqrt{10} \approx P_2(10) = 3 + \frac{1}{6}\cdot 1 - \frac{1}{216}\cdot 1^2 = \underline{\underline{3.1620}}$$

Problem 11

i) We think of y as a function of x and differentiate both sides of the equaton w.r.t. x and get the equation $3x^2 - 4y - 4xy' + 2yy' = 0$, that is $(4x-2y)y' = 3x^2 - 4y$, that is

$$\underline{\underline{y' = \frac{3x^2-4y}{4x-2y}}} \qquad (**)$$

ii) Insert $x = 3$ into the equation for C, that is $y^2 - 12x = -27$, and solve for y. Complete the square and get $(y-6)^2 = -27 + 36 = 9$, that is $y - 6 = \pm 3$, that is $\underline{\underline{y = 9}}$ or $\underline{\underline{y = 3}}$.

We find the slopes for the tangents of C in these points by inserting the x- and y-values into $(**)$.

$$(3,9): \quad y' = \frac{3\cdot 3^2 - 4\cdot 9}{4\cdot 3 - 2\cdot 9} = \frac{-9}{-6} = \underline{\underline{1.5}} \qquad (3,3): \quad y' = \frac{3\cdot 3^2 - 4\cdot 3}{4\cdot 3 - 2\cdot 3} = \frac{15}{6} = \underline{\underline{2.5}}$$

Problem 12

i) We see that $5\ln(x) + 10 \to 0^+$ and $100\ln(x) \to 100\ln(e^{-2}) = -200$ when $x \to (e^{-2})^+$ which means that $f(x) \longrightarrow -\infty$ and especially that $\underline{\underline{x = e^{-2}}}$ is a vertical asymptote. If $x \to \infty$ then $100\ln(x) \to \infty$ and $5\ln(x) + 10 \to \infty$ so we use l'Hôpital's rule to find the limit.

$$\lim_{x\to\infty} f(x) \overset{\text{l'Hôp}}{=} \lim_{x\to\infty} \frac{[100\ln(x)]'}{[5\ln(x)+10]'} = \lim_{x\to\infty} \frac{\frac{100}{x}}{\frac{5}{x}} = \frac{100}{5} = 20$$

Hence the line $\underline{\underline{y = 20}}$ is a horizontal asymptote for $f(x)$.

ii) To find the inverse function expression $g(x)$ we solve the equation $y = f(x)$ for x.

$$y = \frac{100\ln(x)}{5\ln(x)+10} \quad \text{that is} \quad 5y\ln(x) + 10y = 100\ln(x) \quad \text{that is} \quad (5y-100)\ln(x) = -10y$$

which gives

$$\ln(x) = \frac{-10y}{5y-100} = \frac{2y}{20-y} \quad \text{which inserted into } e^x \text{ gives} \quad x = e^{\frac{2y}{20-y}}$$

We change variables and get $\underline{\underline{g(x) = e^{\frac{2x}{20-x}}}}$. We get the domain of definition $D_g = R_f = \underline{\underline{\langle \leftarrow, 20\rangle}}$ from the calculations in (i), and the range $R_g = D_f = \underline{\underline{\langle e^{-2}, \rightarrow\rangle}}$.