# Suggested solution: EBA1180, 03-2023

1) a) $\int_0^7 x^2\sqrt{x}\,dx = \int_0^7 x^{\frac{5}{2}}\,dx = \left[\frac{2}{7}x^{\frac{7}{2}}\right]_{x=0}^{7}$

$= \frac{2}{7}\left(7^3\sqrt{7} - 0\right) = 2\cdot 7^2\sqrt{7} = \underline{\underline{98\sqrt{7}}}$

b) $\int_1^2 \ln(\sqrt{x})\,dx = \int_1^2 \frac{1}{2}\ln x\,dx = \frac{1}{2}\left[x\ln x - x\right]_{x=1}^{2}$

$= \frac{1}{2}(2\ln 2 - 2) - \frac{1}{2}(1\cdot\ln 1 - 1)$

$= \underline{\underline{\ln 2 - \frac{1}{2}}}$

c) $\int_1^2 \frac{6}{x^2-9}\,dx = \int_1^2 \frac{1}{x-3} - \frac{1}{x+3}\,dx$

Partial fractions:

$\frac{6}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$6 = A(x+3) + B(x-3)$

$A = 1,\quad B = -1$

$= \left[\ln|x-3| - \ln|x+3|\right]_{x=1}^{2}$

$= (\ln 1 - \ln 5) - (\ln 2 - \ln 4)$

$= 0 - \ln 5 - \ln 2 + 2\ln 2$

$= \ln 2 - \ln 5 = \underline{\underline{\ln\left(\frac{2}{5}\right)}}$

d) $\int_0^1 \frac{\sqrt{x}}{\sqrt{x}+1}\,dx = \int \frac{\sqrt{x}}{u}\,2\sqrt{x}\,du = \int \frac{2(\sqrt{x})^2}{u}\,du$

Substitution:

$u = \sqrt{x} + 1$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= \int_1^2 \frac{2(u-1)^2}{u}\,du$

$$= \int_1^2 \frac{2(u^2-2u+1)}{u}\,du = \int_1^2 2u - 4 + \frac{2}{u}\,du$$

$$= \Big[u^2 - 4u + 2\ln|u|\Big]_{u=1}^{2}$$

$$= (4-8-2\ln 2) - (1-4+2\ln 1) = \underline{\underline{2\ln 2 - 1}}$$

e) $$\int_{-1}^{0} x\sqrt{-x}\,dx = \int x\sqrt{u}\,(-1)\,du = \int_1^0 -u\sqrt{u}\,(-1)\,du$$

Substitution:
$u = -x$
$du = -dx$

$$= \int_1^0 u^{\frac{3}{2}}\,du = \Big[\tfrac{2}{5}u^{\frac{5}{2}}\Big]_{u=1}^{0}$$

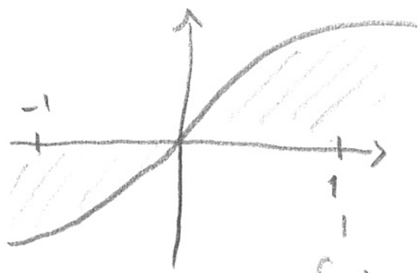
$$= 0 - \frac{2}{5} = \underline{\underline{-\frac{2}{5}}}$$

f) $$\int_{-1}^{1} x\sqrt{|x|}\,dx = \int_{-1}^{0} x\sqrt{-x}\,dx + \int_0^1 x\sqrt{x}\,dx$$

$$= -\frac{2}{5} + \Big[\tfrac{2}{5}x^{\frac{5}{2}}\Big]_{x=0}^{1} = -\frac{5}{2} + \frac{2}{5} - 0 = \underline{\underline{0}}$$

e)

(Can also be seen by symmetry: $f(x) = x\sqrt{|x|}$ gives $f(-x) = -f(x)$, i.e.,




$$\int_{-1}^{1} f(x)\,dx = -A + A = 0$$ )

g) $\int_{1}^{e^2} \frac{\sqrt{\ln x}}{x}\,dx = \int \frac{\sqrt{u}}{x} \cdot x\,du$

Sushtution:

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \int_0^2 \sqrt{u}\,du = \int_0^2 u^{\frac{1}{2}}\,du = \left[\frac{2}{3} u^{\frac{3}{2}}\right]_{u=0}^{2}$$

$$= \frac{2}{3}(2\sqrt{2}) - \frac{2}{3}(0) = \underline{\underline{\frac{4}{3}\sqrt{2}}}$$

2) a) P: $f(x) = a(x-2)^2 + 5$ since $x = 2$ is the axis of symmetry and $y = 5$ is the vertex

$f(2 \pm \sqrt{5}) = a(\pm\sqrt{5})^2 + 5 = 0$ ← a zero

$5a + 5 = 0$

$\underline{a = -1}$

$\Downarrow$

P: $f(x) = 5 - (x-2)^2 = \underline{\underline{1 + 4x - x^2}}$

H: $(x-0)(y-0) = c$ since $x = 0, y = 0$ are asymptotes

$xy = c$

$y = \frac{c}{x}$

H: $g(x) = \frac{c}{x}$

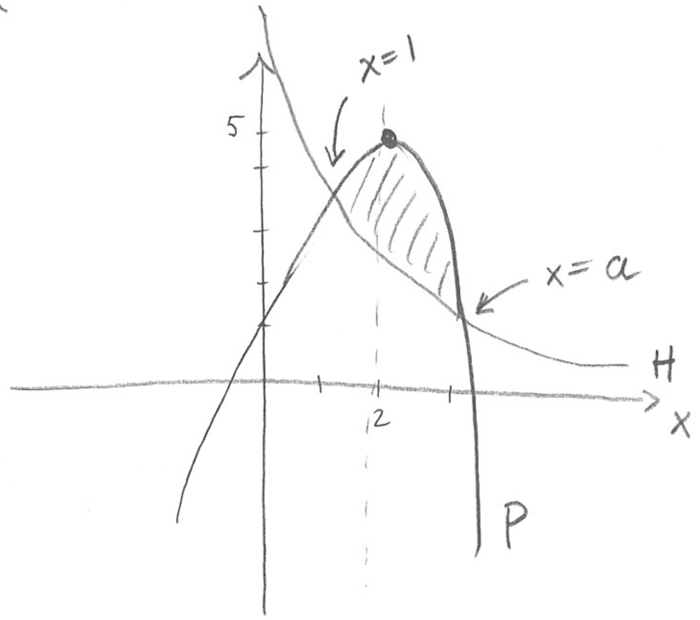
Intersection in $x=1$: $f(1) = g(1)$

$$1+4\cdot 1 - 1^2 = \frac{c}{1}, \quad c = 4$$

$$\Downarrow$$

$$g(x) = \frac{4}{x}$$

b) Area $= \int_1^a f(x) - g(x)\,dx$




Find a:

$$1+4x-x^2 = \frac{4}{x} \quad \Big| \cdot x$$

$$x+4x^2-x^3 = 4$$

$$x^3-4x^2-x+4=0$$ → Polynomial division

$$(x-1)(x^2-3x-4)=0$$

$x=1$ or $x^2-3x-4=0$

$$(x-4)(x+1)=0$$

$\underline{x=4}$, $\underline{x=-1}$ ⟹ $\underline{a=4}$

$$\text{Area} = \int_1^4 1+4x-x^2-\frac{4}{x}\,dx = \left[x+2x^2-\frac{1}{3}x^3-4\ln|x|\right]_{x=1}^4$$

$$= \left(4+2\cdot 16-\frac{1}{3}\cdot 64-4\ln 4\right) - \left(1+2-\frac{1}{3}-4\ln 1\right)$$

$$= 4+32-3-\frac{64}{3}+\frac{1}{3}-4\ln 4$$

$$= 33 - \frac{63}{3} - 4\ln(2^2) = 33 - 21 - 8\ln 2$$

$$= \underline{\underline{12 - 8\ln 2}}$$

3) Total cash flow:

$$\int_0^{25} f(t)\,dt = \int_0^{25} 100\,e^{\sqrt{t}}\,dt = \int 100\,e^u\,2\sqrt{t}\,du$$

Substitution:

$u = \sqrt{t}$

$du = \frac{1}{2\sqrt{t}}\,dt$

$$= 200\int_0^5 e^u\,u\,du = 200\left[ue^u - e^u\right]_{u=0}^{5}$$

Partial integration

$$= 200\,(5e^5 - e^5) - 200\,(0-1) = 200\,e^5(4) + 200$$

$$= \underline{\underline{800e^5 + 200}}$$

Expression for the present value:

$$\int_0^{25} f(t)\,e^{-rt}\,dt = \underline{\underline{\int_0^{25} 100\,e^{\sqrt{t}}\,e^{-rt}\,dt}}$$

4) a) $$\left[\begin{array}{cccc|c} 2 & 1 & 2 & -3 & 4 \\ 3 & -1 & 8 & 2 & 7 \\ 5 & 5 & 0 & -17 & 12 \end{array}\right] \begin{array}{c} -1 \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} -1 & 2 & -6 & -5 & -3 \\ 3 & -1 & 8 & 2 & 7 \\ 5 & 5 & 0 & -17 & 12 \end{array}\right] \begin{array}{c} \\ 3 \\ 5 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} -1 & 2 & -6 & -5 & -3 \\ 0 & 5 & -10 & -13 & -2 \\ 0 & 15 & -30 & -42 & -3 \end{array}\right] \begin{array}{c} \\ \\ -3 \end{array} \sim \left[\begin{array}{cccc|c} -1 & 2 & -6 & -5 & -3 \\ 0 & 5 & -10 & -13 & -2 \\ 0 & 0 & 0 & -3 & 3 \end{array}\right]$$

$z$ is free

$-3w = 3$, so $\underline{w = -1}$

$5y = 10z + 13(-1) - 2 = 10z - 15$, so $y = \underline{2z - 3}$

$-x = -2(2z-3) + 6z + 5\cdot(-1) - 3 = 2z - 2$, so $x = \underline{-2z+2}$

Infinitely many solutions:

$(x, y, z, w) = \underline{(-2z+2,\ 2z-3,\ z,\ -1)}$ with $z$ free.

b) $$\left[\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 3 & 1 & 2 & 4 \\ 2 & -1 & 4 & 3 \\ 4 & 5 & 1 & 13 \end{array}\right] \begin{array}{c} -3 \\ -2 \\ \\ -4 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 0 & -8 & 11 & -2 \\ 0 & -7 & 10 & -1 \\ 0 & -7 & 13 & 5 \end{array}\right] \begin{array}{c} \\ -1 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -7 & 10 & -1 \\ 0 & -7 & 13 & 5 \end{array}\right] \begin{array}{c} \\ -7 \\ \\ -7 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 6 & 12 \end{array}\right] \begin{array}{c} \\ \\ \\ -2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

One solution:

$3z = 6$, so $\underline{z = 2}$

$-y = -(2) - 1 = -3$, so $\underline{y = 3}$

$x = -3(3) + 3(2) + 2 = -1$, so $\underline{x = -1}$

Solution: $\underline{\underline{(x, y, z) = (-1, 3, 2)}}$

5) a) $$\begin{vmatrix} a & 2 & 3 \\ 2 & a & 3 \\ 2 & 3 & a \end{vmatrix} = a(a^2 - 9) - 2(2a - 6) + 3(6 - 2a)$$

$$= a(a-3)(a+3) - 4(a-3) - 6(a-3)$$

$$= (a-3)\big(a(a+3) - 4 - 6\big)$$

$$= (a-3)(a^2 + 3a - 10)$$

$$= \underline{\underline{(a-3)(a-2)(a+5)}}$$

$$= \underline{\underline{a^3 - 19a + 30}}$$

Note that $a = 3$ makes rows 2 and 3 alike, i.e., $|A| = 0$, so $(a-3)$ must be a factor in $|A|$

b) $\underline{a = 0}$: $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$, $|A| = (-2)(-6) + 3 \cdot 6 = 12 + 18 = 30 \neq 0$, so $A^{-1}$ exists.

$$A^{-1} = \frac{1}{30}\begin{bmatrix} -9 & 6 & 6 \\ 9 & -6 & 4 \\ 6 & 6 & -4 \end{bmatrix}^T = \frac{1}{30}\begin{bmatrix} -9 & 9 & 6 \\ 6 & -6 & 6 \\ 6 & 4 & -4 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b} = \frac{1}{30}\begin{bmatrix} -9 & 9 & 6 \\ 6 & -6 & 6 \\ 6 & 4 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{30}\begin{bmatrix} -6 \\ -6 \\ 14 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{5} \\ \frac{7}{15} \end{bmatrix}$$

c) $A\vec{x} = \vec{b}$ has a unique solution $\Leftrightarrow$ $|A| \neq 0$

Know: $|A| = 0$ for $a = 2, 3, -5$ $\Rightarrow$ Unique solution for

$a \neq 2, 3, -5.$

d) Potential values of $a$ with infinitely many solutions:

$a = 2, 3, -5.$

$\underline{a=2}$: $\left[\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 2 & 2 & 3 & 1 \\ 2 & 3 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Infinitely many solutions: $z$ free, $y = z - 2$,

$2x = -2(z-2) - 3z + 1 = -5z + 5 \Rightarrow x = \frac{-5z}{2} + \frac{5}{2}$

solution: $(x,y,z) = \left(-\frac{5z}{2} + \frac{5}{2},\ z-2,\ z\right)$, $z$ free.

$\underline{a=3}$: $\left[\begin{array}{ccc|c} 3 & 2 & 3 & 1 \\ 2 & 3 & 3 & 1 \\ 2 & 3 & 3 & -1 \end{array}\right] \Rightarrow$ No solution.

$\underline{a=-5}$: $\left[\begin{array}{ccc|c} -5 & 2 & 3 & 1 \\ 2 & -5 & 3 & 1 \\ 2 & 3 & -5 & -1 \end{array}\right] 2 \sim \left[\begin{array}{ccc|c} -1 & -8 & 9 & 3 \\ 2 & -5 & 3 & 1 \\ 2 & 3 & -5 & -1 \end{array}\right] 2 \; 2$

$\sim \left[\begin{array}{ccc|c} -1 & -8 & 9 & 3 \\ 0 & -21 & 21 & 7 \\ 0 & -13 & 13 & 5 \end{array}\right] -\frac{13}{21} \sim \left[\begin{array}{ccc|c} -1 & -8 & 9 & 3 \\ 0 & -21 & 21 & 7 \\ 0 & 0 & 0 & \underbrace{5-\frac{13}{3}}_{\neq 0} \end{array}\right]$

$\Rightarrow$ No solution.

6) a) $\left[\begin{array}{cc|c} 5 & 3 & 1 \\ 4 & 1 & 5 \\ 7 & 2 & 8 \end{array}\right] -1 \sim \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 4 & 1 & 5 \\ 7 & 2 & 8 \end{array}\right] -4 \; -7$

$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$

$\sim \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & -7 & 21 \\ 0 & -12 & 36 \end{array}\right] -\frac{12}{7} \sim \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & -7 & 21 \\ 0 & 0 & 0 \end{array}\right]$

One solution.

$-7y = 21 \Rightarrow \underline{y=-3}$

$x + 2(-3) = -4 \Rightarrow x = 6-4 = 2$, so $\underline{\vec{v}_3 = 2\vec{v}_1 - 3\vec{v}_2}$

b)

$$\left[\begin{array}{cccc|c} 5 & 3 & 1 & 3 & a \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \end{array}\right] \begin{array}{l} \\ -1 \\ \\ \end{array} \qquad \vec{w} = x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 + w\vec{v}_4$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \end{array}\right] \begin{array}{l} \\ -4 \\ -7 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & b-4(a-b) \\ 0 & -12 & 36 & 48 & c-7(a-b) \end{array}\right] \begin{array}{l} \\ \\ -\frac{12}{7} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & -4a+5b \\ 0 & 0 & 0 & 0 & \underbrace{c-7(a-b)-\frac{12}{7}(b-4(a-b))}_{(*)} \end{array}\right]$$

$\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$

$\Leftrightarrow$ the linear system is consistent

$\Leftrightarrow$ $\underline{(*) = 0:}$

$$c - 7(a-b) - \frac{12}{7}(b - 4(a-b)) = 0 \quad \Big| \cdot 7$$

$$7c - 49(a-b) - 12b + 48(a-b) = 0$$

$$7c - 12b - (a-b) = 0$$

$$-a - 11b + 7c = 0 \quad \Big| \cdot (-1)$$

$$a + 11b - 7c = 0$$

Conclusion: $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \iff a + 11b - 7c = 0.$

c) $\vec{w} \perp \vec{v}_2 \iff \vec{w} \circ \vec{v}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 0$

$\iff 3a + b + 2c = 0$, $b, c$ free and

$$\frac{3a}{3} = \frac{-b-2c}{3}$$

$$a = -\frac{b}{3} - \frac{2c}{3}$$

Solution: $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{b}{3} - \frac{2c}{3} \\ b \\ c \end{bmatrix} = \frac{b}{3}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \frac{c}{3}\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

$$= s\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

7.) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$AX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$\underline{AX = XA:}$ $\left.\begin{array}{l} c = b \\ d = a \\ a = d \\ b = c \end{array}\right\} \Rightarrow c, d$ free, $a = d$, $b = c$

$(a, b, c, d) = (d, c, c, d) = c(0,1,1,0) + d(1,0,0,1)$, $c, d$ free.

Conclusion:

$X = c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with $c, d$ free (i.e., all linear combinations of $A$ and $I$)

8) a) $\left.\begin{array}{l} f'_x = 2x - 4y - 4 = 0 \\ f'_y = -4x + 10y + 4 = 0 \end{array}\right\}$ $\begin{array}{l} 2x - 4y = 4 \\ -4x + 10y = -4 \end{array}$

A linear system:

$$\left[\begin{array}{cc|c} 2 & -4 & 4 \\ -4 & 10 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -4 & 4 \\ 0 & 2 & 4 \end{array}\right]$$

$\Rightarrow$ $2y = 4$, so $\underline{y = 2}$

$2x - 4y = 4$, so $2x = 4 + 4 \cdot 2 = 12$

$\underline{x = 6}$

One stationary point: $\underline{(x, y) = (6, 2)}$

$$H(f) = \begin{bmatrix} 2 & -4 \\ -4 & 10 \end{bmatrix} \qquad \det H(f) = 2 \cdot 10 - (-4)(-4)$$

$$= 20 - 16 = 4 > 0$$

$$\operatorname{tr}(H)(f) = 2 + 10 = 12 > 0$$

Hence, $\underline{(6, 2) \text{ is a local minimum}}$ from the second derivative test.

$$f(6, 2) = 6^2 - 4 \cdot 6 \cdot 2 + 5 \cdot 2^2 - 4 \cdot 6 + 4 \cdot 2 + 1$$

$$= 36 - 48 + 20 - 24 + 8 + 1$$

$$= \underline{-7}$$

b) Make a change of variables $\begin{cases} u = x - 6 \\ v = y - 2 \end{cases}$ s.t. the stationary point becomes $u = v = 0$. This gives $x = u + 6$, $y = v + 2$ and:

$$f(u, v) = (u+6)^2 - 4(u+6)(v+2) + 5(v+2)^2$$
$$- 4(u+6) + 4(v+2) + 1$$

$$= u^2 + 12u + 36 - 4(uv + 2u + 6v + 12) + 5(v^2 + 4v + 4)$$
$$- 4u - 24 + 4v + 8 + 1$$

$$= u^2 - 4uv + 5v^2 + \cancel{12u} - \cancel{8u} - \cancel{24v} + \cancel{20v} - \cancel{4u} + \cancel{4v}$$
$$+ 36 - 48 + 20 - 24 + 8 + 1$$

$$= u^2 - 4uv + 5v^2 - 7 = (u - 2v)^2 + v^2 - 7 \geq -7$$

Hence, $(u,v) = (0,0)$ or $(x,y) = (6,2)$, is a global minimum and $f_{min} = -7$.

The function $f$ has no (local or global) max.

9) a) $f'_x = 2xy^3 = 0 \Rightarrow x = 0$ or $y = 0$

$f'_y = x^2 \cdot 3y^2 + 2y - 2 = 0$

| $x=0$ | $y=0$ |
|---|---|
| $2y - 2 = 0$ | $-2 = 0$ |
| $y = 1$ | Never true $\Rightarrow$ No point. |
| $(x,y) = \underline{(0,1)}$ | |

<u>One stationary point:</u> $(x,y) = (0,1)$.

$$H(f) = \begin{bmatrix} 2y^3 & 2x \cdot 3y^2 \\ 2x \cdot 3y^2 & x^2 6y + 2 \end{bmatrix}$$

$$H(f)(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det H(f)(0,1) = 4 - 0 = 4 > 0$

$\operatorname{tr} H(f)(0,1) = 2 + 2 = 4 > 0$

$\Rightarrow$ <u>$(x,y) = (0,1)$ is a local minimum</u> from the second derivative test.

b) <u>Maximum:</u> None.

<u>Minimum:</u> $f(0,1) = 1-2+1 = 0$, local min.

$f(1,-3) = 1\cdot(-3)^3 + (-3)^2 - 2(-3) + 1 = -27+9+6+1$

$= -12 < 0 \Rightarrow$ <u>No minimum</u>,

since $f(0,1)=0$ is not a global min.

(Actually: $f(1,y) = y^3 + y^2 - 2y + 1 \to -\infty$ when $y \to -\infty$)