Optimization: More about the second derivative test

EBA 1180
Lecture 18
42
S26

If (x^*, y^*) is a stationary point with

$$\mathcal{H}(f)(x^*, y^*) = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

$A \to f''_{xx}$, $B \to f''_{xy}$, $B \to f''_{yx}$, $C \to f''_{yy}$

Then: ① If $\underbrace{AC - B^2}_{\text{determinant of Hessian}} > 0$, then $AC > B^2 \geq 0$

> 0: Local max. or local min.

$\Rightarrow AC > 0$. But then,

$+ \cdot + = +$

i) $A, C > 0$: $\underbrace{A + C}_{\text{trace of Hessian}} > 0 \Rightarrow$ Local min. from 2nd derivative test.

$- \cdot - = +$

ii) $A, C < 0$: $A + C < 0 \Rightarrow$ Local max. from 2nd derivative test.

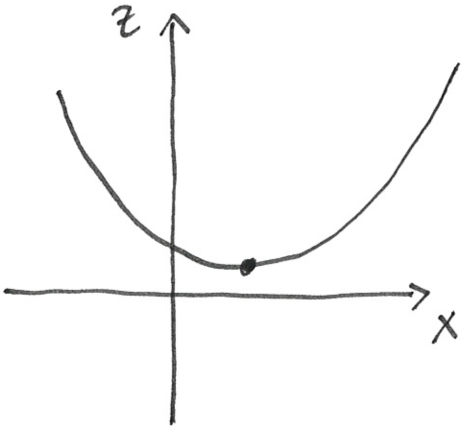
Possible cases: i) $A, C > 0$: Local min. ($A \to f''_{xx}$, $C \to f''_{yy}$)

ii) $A, C < 0$: Local max.

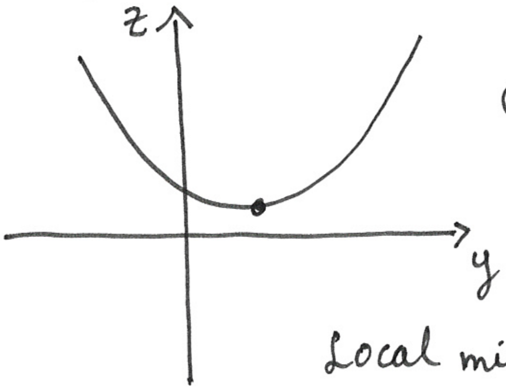
Recall: 1 var. functions:
2nd order derivative ≥ 0; convex
— " — ≤ 0; concave

Graphically: Cuts of graph of $z = f(x,y)$

i) $A, C > 0$: Locally convex cuts



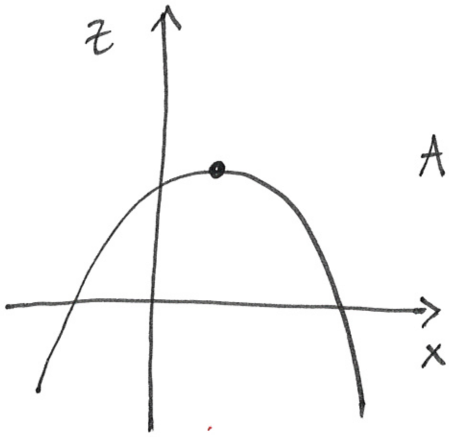
$A = f''_{xx} > 0$



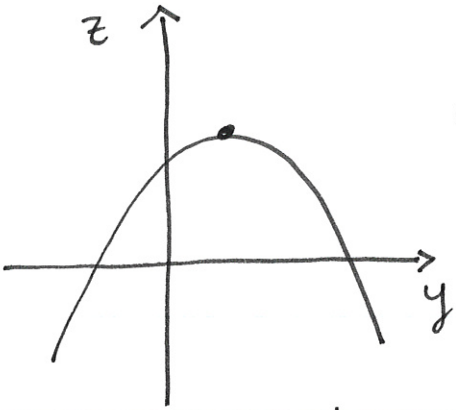
$C = f''_{yy} > 0$

Local minimum; Locally convex

ii) $A, C < 0$: Locally concave cuts



$A = f''_{xx} < 0$



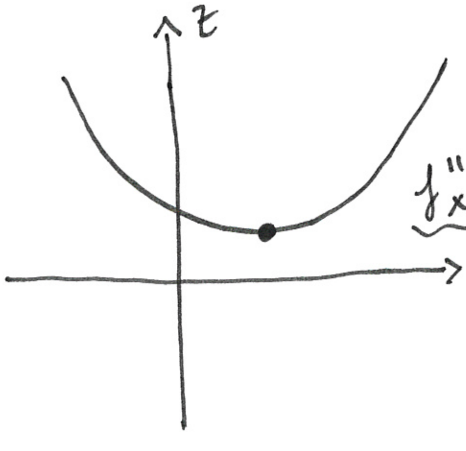
$C = f''_{yy} < 0$

Local maximum; Locally concave

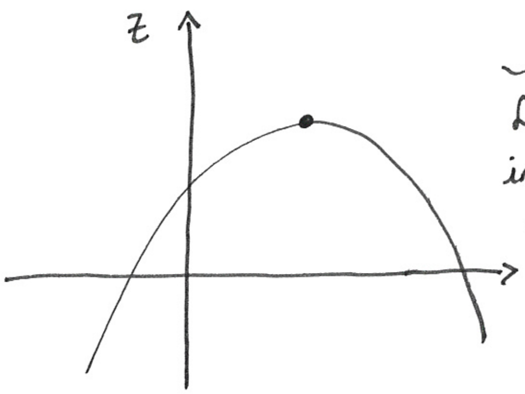
② If $\underbrace{AC - B^2}_{\text{determinant of the Hessian}} \underbrace{< 0}_{\text{2nd deriv. test: Saddle pt.}}$: $AC < B^2$

A typical case:

$\underset{f''_{xx}}{A} > 0, \quad \underset{f''_{yy}}{C} < 0$



$f''_{xx} > 0$: Local min. in x-direction. Locally convex in x-dir.



$f''_{yy} < 0$: Local max in y-dir. Locally concave in y-dir.

Course paper 2023

1) g) $$\int_1^{e^2} \frac{\sqrt{\ln(x)}}{x}\,dx = \int \frac{\sqrt{u}}{\cancel{x}}\,\cancel{x}\,du$$

Substitution:

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$x = 1 \Rightarrow u = \ln x = \ln 1 = 0$

$x = e^2 \Rightarrow u = \ln x = \ln e^2 = 2$

$$= \int_0^2 \sqrt{u}\,du$$

$$= \int_0^2 u^{\frac{1}{2}}\,du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=0}^{2}$$

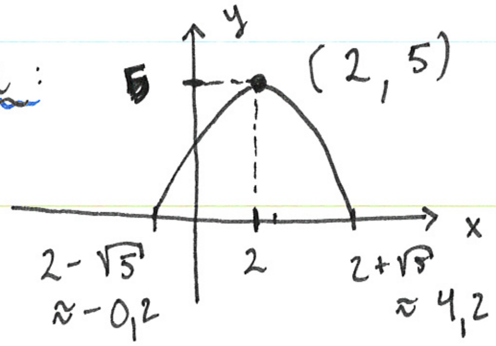
$$= \frac{2}{3}\Big(\underbrace{2^{\frac{3}{2}}}_{2\cdot 2^{\frac{1}{2}}} - \underbrace{0^{\frac{3}{2}}}_{0}\Big) = \frac{2}{3}\left(2\sqrt{2} - 0\right)$$

$$= \frac{4}{3}\sqrt{2} = \underline{\underline{\frac{4\sqrt{2}}{3}}}$$

START: 11.03

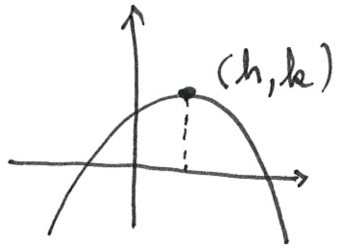
2) Parabola:

SKETCH:



Vertex form:

$y = a(x-h)^2 + k$



P:

$f(x) = y = a(x-2)^2 + 5$

Parabola intersects x-axis in $2 \pm \sqrt{5}$:

$$f(2 \pm \sqrt{5}) = 0$$

From P: $$a(\pm\sqrt{5})^2 + 5 = 0$$

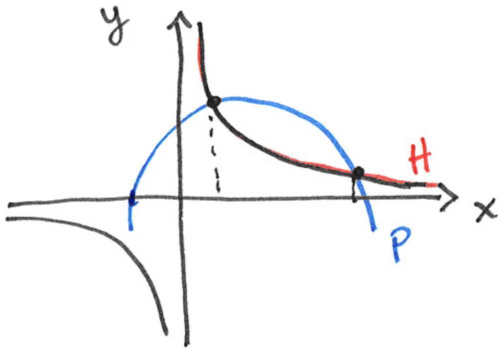
$$5a = -5 \Rightarrow \underline{a = -1}$$

Parabola: $f(x) = 5-(x-2)^2 = \underline{1+4x-x^2}$

Hyperbola:

SKETCH:

Asymptotes at $x=0$, $y=0$



Intersects P in $x=1$

General formula for hyperbola:

$$(x-\underset{\text{x-asymptote}}{0})(y-\underset{\text{y-asymptote}}{0}) = c$$

$$xy = c$$

$$y = \frac{c}{x} \qquad (x \neq 0 \text{ since } c \neq 0)$$

H: $g(x) = \frac{c}{x}$, what is c?

H and P intersect at $x=1$: P = H at $x=1$

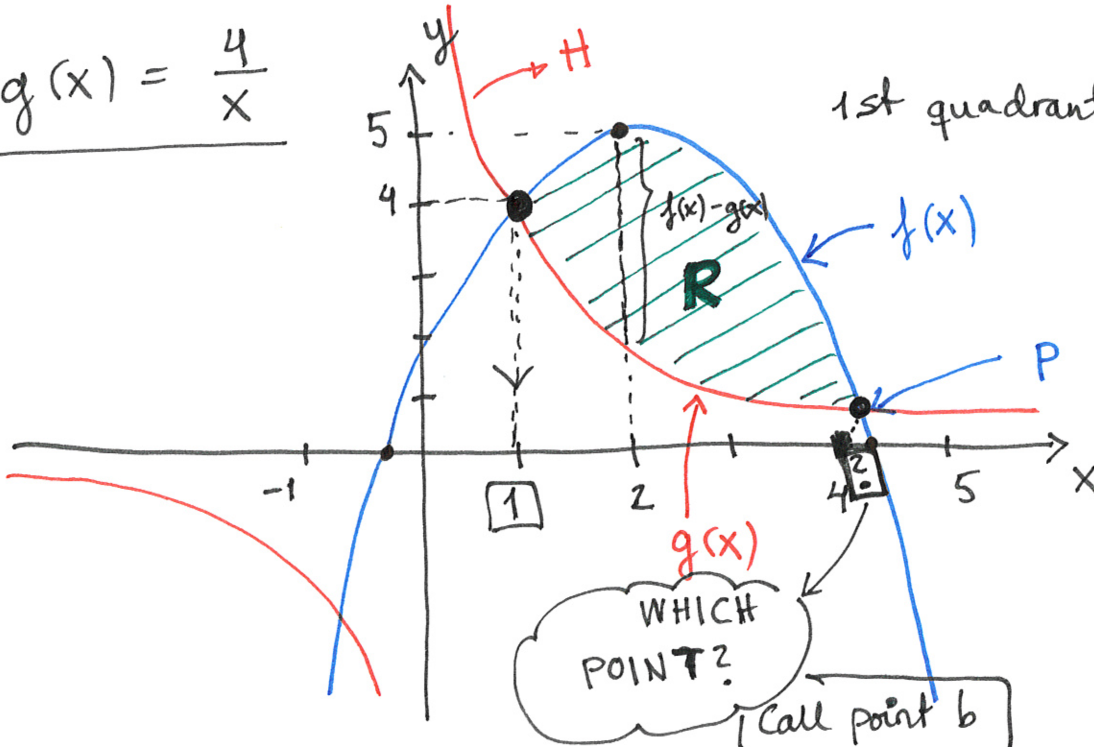
$$f(1) = g(1)$$

MINUS!

$$1+4\cdot 1 - 1^2 = \frac{c}{1} \Rightarrow \underline{c=4}$$

H: $\underline{g(x) = \frac{4}{x}}$

1st quadrant



Need to find b to get upper integration bound.

Area of R :
$$\int_1^b f(x) - g(x) \, dx$$

Find b : $f(x) = g(x)$

$$1 + 4x - x^2 = \frac{4}{x} \quad \Big| \cdot x$$

$$x + 4x^2 - x^3 = 4$$

$$x^3 - 4x^2 - x + 4 = 0 \;\rightarrow$$

Know : f and g intersect at x=1. So x=1 is a solution

Polynomial division :

$$\begin{array}{l} (x^3 - 4x^2 - x + 4) : (x-1) = \underbrace{x^2 + \ldots}_{} \\ -(x^3 - x^2) \\ \hline \quad \ldots \end{array}$$

$\boxed{\text{DIY}}$

$$x^3 - 4x^2 - x + 4 = (x-1)\cdot(x^2 - 3x - 4) = 0$$

$\underline{x = 1}$ or $x^2 - 3x - 4 = 0$

$$(x-4)(x+1) = 0$$

$$\underline{x = 4}, \; \underline{x = -1}$$

$\Rightarrow$ $\underline{b = 4}$

From figure

Area of R: $\displaystyle\int_1^4 \underbrace{1+4x-x^2}_{f(x)} - \underbrace{\frac{4}{x}}_{g(x)}\,dx$

$$= \left[x + 2x^2 - \frac{1}{3}x^3 - 4\ln|x|\right]_{x=1}^{4}$$

$= \ldots$ plug in numbers $\ldots = 12 - 8\ln 2$

Solve:

7) $AX = XA$

$\underbrace{\text{X must be } 2\times 2 \text{ for AX and XA to be}}$ defined

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$AX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Want: $AX = XA$

$$\left.\begin{aligned} c &= b \\ d &= a \\ a &= d \\ b &= c \end{aligned}\right\} \quad a = d \text{ and } b = c, \quad c, d \text{ free}$$

$$(a, b, c, d) = (d, c, c, d) = c(0,1,1,0) + d(1,0,0,1)$$

where c and d are free.

Conclusion:

$$X = c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d & c \\ c & d \end{bmatrix}$$ where c and d are free.

Rest of solutions:
Itslearning.