Optimization of functions in two variables

EBA 1180 Lect 16/40 S26

Def: (The Hessian)

The Hessian of $f(x,y)$ is the 2×2 matrix

$$\mathcal{H}(f)(x,y) := \begin{bmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{bmatrix}$$

Def (Max/min):

i) (x^*, y^*) is a maximal point/maximizer for f if

$$f(x^*, y^*) \geq f(x,y) \text{ for all } (x,y) \in D_f.$$

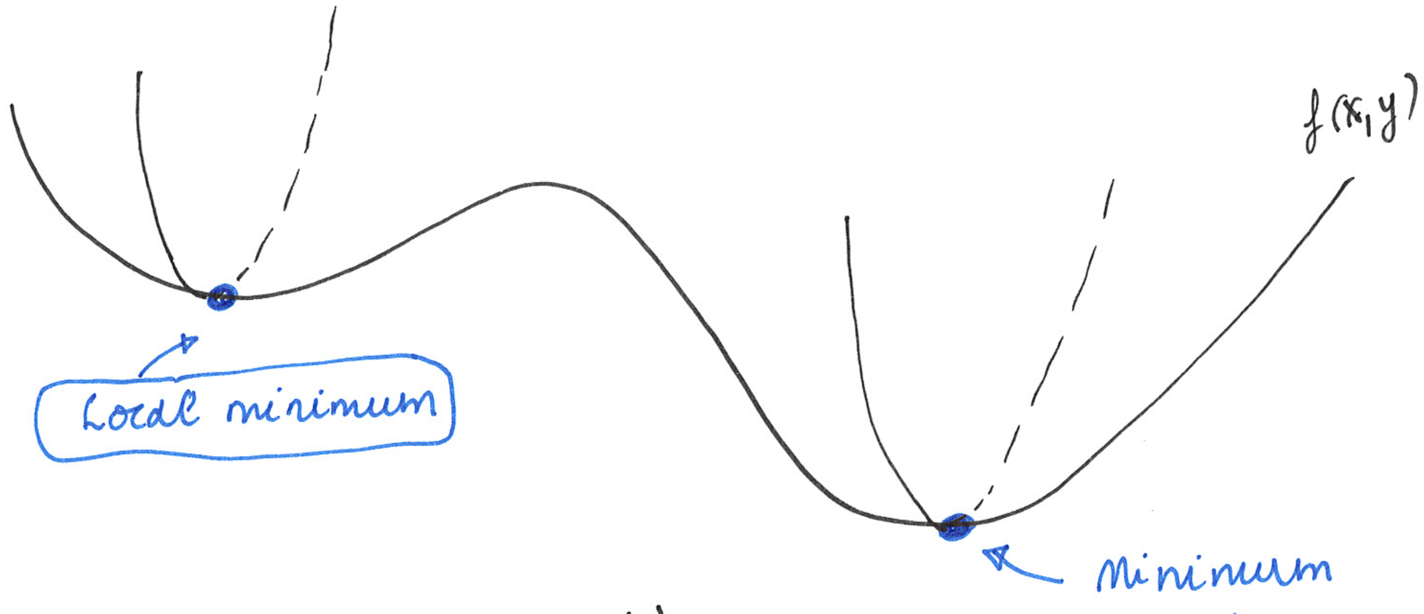
ii) (x^*, y^*) is a local max. for f if $f(x^*, y^*) \geq f(x,y)$ for all (x,y) close to (x^*, y^*).

iii) (x^*, y^*) is a minimum point/minimizer for f if

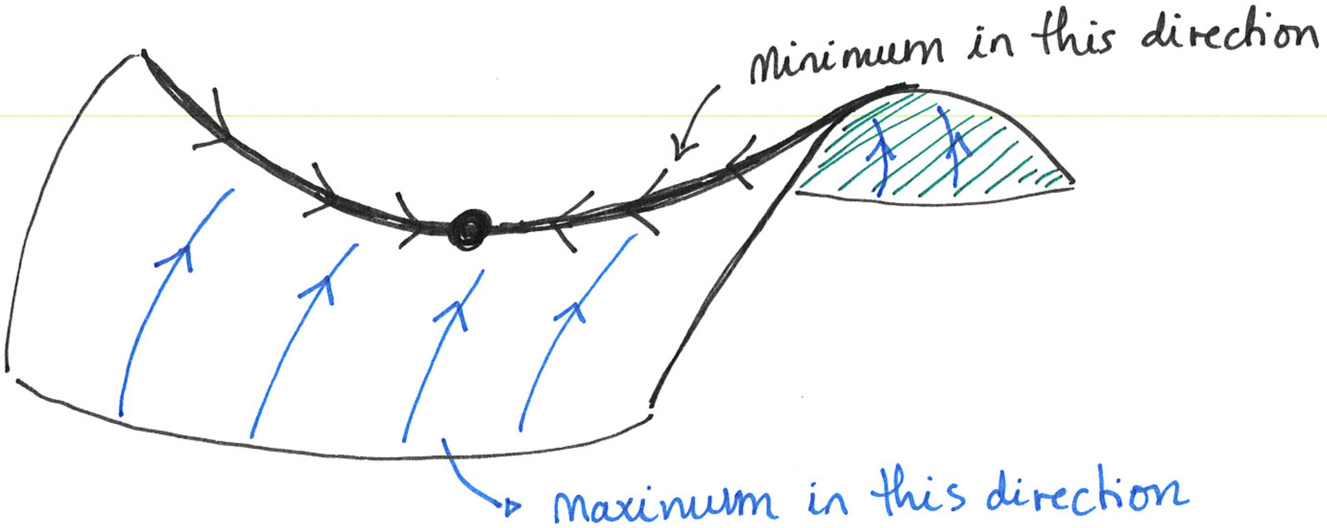
$$f(x^*, y^*) \leq f(x,y) \text{ for all } (x,y) \in D_f.$$

iv) (x^*, y^*) is a local minimum for f if

$f(x^*, y^*) \leq f(x, y)$ for all (x, y) close to (x^*, y^*).



v) A stationary point (x^*, y^*) of f which is neither a local max. nor a local min. is a saddle point.

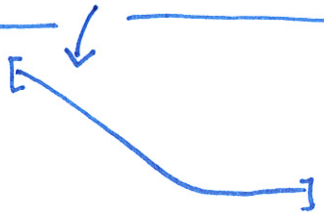


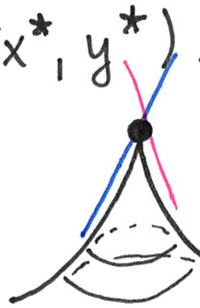
KEY RESULT: If (x^*, y^*) is a max/min. point for f, then we have either:

i) (x^*, y^*) is a stationary point for f: $f'_x = f'_y = 0$ at (x^*, y^*).

ii) Either f'_x or f'_y is not defined at (x^*, y^*).

iii) (x^*, y^*) is a boundary point of D_f.





Cusp: Derivative not def: Jumps dep. on going from left or right.

The second derivative test

Result: If (x^*, y^*) is a stationary point for f, we get:

$$\mathcal{H}(f)(x^*, y^*) = \begin{bmatrix} f''_{xx}(x^*, y^*) & f''_{xy}(x^*, y^*) \\ f''_{yx}(x^*, y^*) & f''_{yy}(x^*, y^*) \end{bmatrix}$$

If cross derivatives exist and are cont.

$$= \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

$AC - B^2$

We have:

1) If $\det \mathcal{H}(f)(x^*, y^*) > 0$ and $\operatorname{tr} \mathcal{H}(f)(x^*, y^*) > 0$, then (x^*, y^*) is a local min.

"Trace": $A + C$

2) If $\det \mathcal{H}(f)(x^*, y^*) > 0$ and $\operatorname{tr} \mathcal{H}(f)(x^*, y^*) < 0$, then (x^*, y^*) is a local max.

3) If $\det \mathcal{H}(f)(x^*, y^*) < 0$, then (x^*, y^*) is a saddle point.

NOTE: If $\det \mathcal{H}(f)(x^*, y^*) = 0$, the test is inconclusive.

START 11.02

Ex:

$$f(x,y) = \underbrace{x^3 + 3xy + y^3}_{\substack{\text{Partial derivatives defined} \\ \text{everywhere : No points} \\ \text{where either } f'_x \text{ or } f'_y \\ \text{are not defined}}}, \quad \underbrace{D_f = \mathbb{R}^2}_{\substack{\text{No boundary points} \\ \text{to consider}}}$$

Stationary points: Also the candidate points for max/min

FOC: First order conditions

$$\begin{cases} f'_x = 3x^2 + 3y = 0 \\ f'_y = 3x + 3y^2 = 0 \end{cases}$$

$$\begin{cases} x^2 + y = 0 \quad \Rightarrow \quad y = -x^2 \quad (*) \\ x + y^2 = 0 \quad \longrightarrow \quad x + (-x^2)^2 = 0 \end{cases}$$

$$x + x^4 = 0$$

$$x \cdot (1 + x^3) = 0$$

$\underline{x=0}$:

From (*): $y = -0^2 = \underline{\underline{0}}$

$\underline{1+x^3=0}$:

$x^3 = -1$

$x = \sqrt[3]{-1} = \underline{\underline{-1}}$

From (*):

$y = -(-1)^2 = \underline{\underline{-1}}$

The stationary points are: $(0,0)$ and $(-1,-1)$.

Since there are no boundary points or points where either partial derivative is not defined, the stationary points are also all of the candidate points.

Function values in candidate points:

$$f(0,0) = 0^3 + 3\cdot 0\cdot 0 + 0^3 = \underline{\underline{0}}$$

$$f(-1,-1) = (-1)^3 + 3\cdot(-1)\cdot(-1) + (-1)^3 = -1+3-1 = \underline{\underline{1}}$$

Classify candidate points by using the second derivative test

Hessian:

$$\mathcal{H}(f) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} 6x & 3 \\ 3 & 6y \end{bmatrix}$$

Candidate point $(0,0)$:

$$\mathcal{H}(f)(0,0) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \Rightarrow \det \mathcal{H}(f)(0,0) = 0\cdot 0 - 3\cdot 3 = -9 < 0$$

$\Rightarrow$ $(0,0)$ is a saddle point for f.

Candidate point $(-1,-1)$:

$$\mathcal{H}(f)(-1,-1) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$$

$$\det \mathcal{H}(f)(-1,-1) = 36 - 9 = 27 > 0$$

Since >0: Need to compute trace to classify

$$\operatorname{tr} \mathcal{H}(f)(-1,-1) = -6 + (-6) = -12 < 0$$

$$\Downarrow$$

$(-1,-1)$ is a local max. for f.

Global max/min.?

Conclusion: $f(x,y) = x^3 + 3xy + y^3$ has no minimum. But $f(x,y)$ has a local maximum $f(-1,-1) = 1$ at $(-1,-1)$.

No candidates

NOTE: $f(10,10) = 10^3 + 3\cdot 10\cdot 10 + 10^3 = 2300 > \underbrace{1}_{\text{value in local max.}}$

$\Rightarrow$ $(-1,-1)$ is not a global max. for f.

$\Rightarrow$ f has no global max.

Tangents of level curves

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

Level curves: All (x,y) s.t. $f(x,y) = c$:

$$x^2 - 2x + y^2 + 4y = c$$

TRICK: Complete the squares:

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) = c + 1 + 4$$

$$(x-1)^2 + (y+2)^2 = c+5 \quad (*)$$

3 cases:

i) $c+5>0$:

So $c > -5$

$\Rightarrow$ Level curve is a circle with center in $(1,-2)$ and $r = \sqrt{c+5}$

ii) $c+5=0$:

So $c=-5$

$\Rightarrow$ Level "curve" is $(1,-2)$. WHY?

Need $\begin{cases} x-1=0 \\ y+2=0 \end{cases}$ because sum of squares is 0 $\Rightarrow$ $x=1$, $y=-2$

iii) $c+5<0$:

So $c<-5$

$\Rightarrow$ $(*)$ never holds (sum of squares never neg.)

$\Rightarrow$ No level curve, i.e. f never attains a value $c=-5$