The transpose

$$\underset{m \times n}{A} \xrightarrow[\boxed{\text{TRANSPOSE}}]{} \underset{n \times m}{A^T}$$

"A transpose" or "transpose of A"

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$$

2×3 $\qquad$ 3×2

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 1 & 7 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 7 \\ 0 & 4 & 3 \end{bmatrix}$$

3×3 $\qquad$ 3×3

Main diagonal is preserved when transposing square matrices

Def (Symmetric matrix): A is a symmetric matrix if

$$A = A^T$$

Ex:

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{bmatrix} = B$$

So $B = B^T$, hence B is symmetric.

Rules for matrix algebra

"Normal"

→ $A + B = B + A$

→ $A \cdot (B + C) = A \cdot B + A \cdot C$

→ $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

"Not normal"

→ $AB \neq BA$

→ $(A+B)^2 = (A+B)(A+B)$

$= A^2 + AB + BA + B^2$

$\neq A^2 + 2AB + B^2$

(in general)

Determinants:

→ $|A \cdot B| = |A| \cdot |B|$

→ $|cA| = c^n |A|$ for A $n \times n$

→ $|A^T| = |A|$

All can be proved, but none shall

Transpose:

→ $(A^T)^T = A$

→ $(AB)^T = B^T A^T$

(To be determined in model)

Variables: x, y, z

Parameter: r

Given from outside

Ex:

3x3 linear system

$$\begin{cases} rx + 2y - z = 3 \\ x + (r+1)y - z = 3 \\ -x - 2y + rz = 1-r \end{cases}$$

$$|A| = \begin{vmatrix} r & 2 & -1 \\ 1 & r+1 & -1 \\ -1 & -2 & r \end{vmatrix}$$

coefficient matrix

Sign of cofactor expansion 3x3:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= r \begin{vmatrix} r+1 & -1 \\ -2 & r \end{vmatrix}$$

$$- 2 \begin{vmatrix} 1 & -1 \\ -1 & r \end{vmatrix} - 1 \begin{vmatrix} 1 & r+1 \\ -1 & -2 \end{vmatrix}$$

$$= r\left((r+1)r - 2\right) - 2(r-1) - 1(-2 + r+1)$$

$$= r(r^2 + r - 2) - 2r + 2 + 2 - r - 1$$

$$= r(r+2)(r-1) - 3r + 3$$

abc-formula etc. to factorize

$$= r(r+2)\underline{(r-1)} - 3\underline{(r-1)}$$

$$= (r-1)\left(r(r+2) - 3\right)$$

$$= (r-1)(r^2 + 2r - 3)$$

$$= (r-1)(r-1)(r+3) = \underline{\underline{(r-1)^2(r+3)}}$$

2 cases:

1) $|A| = 0$: No or infinitely many solutions.

Which one?

$$(r-1)^2(r+3) = 0$$

$r = 1$, $r = -3$

Check for both $r=1$ and $r=-3$:

i) $r=1$: Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 2 & -1 & 3 \\ -1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon form!

$\rightarrow$ $0 = 3$ NOT TRUE!

$\Rightarrow$ No solutions for $r=1$

ii) $r=-3$: DIY

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & 3 \\ 1 & -2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right] \sim \; \dots \; \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

$0 = -5$ NOT TRUE!

No solutions for $r=-3$

2) $|A| \neq 0$:

$r \neq 1$

$r \neq -3$

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2) $|A| \neq 0$: $\quad (r-1)^2(r+3) \neq 0$

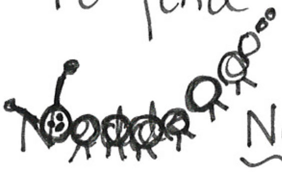
$r \neq 1, \; r \neq -3 \Rightarrow$ Unique solution

Find this solution via Cramer's rule : Easier than Gaussian elimination because of parameters

Know : $|A| = (r-1)^2(r+3)$

To find x (x_1 in formulation of Cramer's rule):

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$



Need : $$|A_1(\vec{b})| = \begin{vmatrix} 3 & 2 & -1 \\ 3 & r+1 & -1 \\ 1-r & -2 & r \end{vmatrix}$$

$\vec{b} = \begin{bmatrix} 3 \\ 3 \\ 1-r \end{bmatrix}$

$$= 3\begin{vmatrix} r+1 & -1 \\ -2 & r \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 1-r & r \end{vmatrix} - 1\begin{vmatrix} 3 & r+1 \\ 1-r & -2 \end{vmatrix}$$

$$= \ldots = \underline{2r^2 - r - 1}$$

From Cramer's rule : $$x = \frac{|A_1(\vec{b})|}{|A|} = \frac{2r^2 - r - 1}{(r-1)^2(r+3)}$$

abc-formula

$$= \frac{(2r+1)\cancel{(r-1)}}{(r-1)^{\cancel{2}}(r+3)} = \frac{2r+1}{(r-1)(r+3)}, \quad r \neq 1, -3$$

Can find $\overset{x_2}{y}$ and $\overset{x_3}{z}$ similarly.

$$|A_2(\vec{b})| \qquad |A_3(\vec{b})|$$

Vector equations

$$x \underbrace{\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}}_{\vec{v}_1} + y \underbrace{\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}_{\vec{v}_2} + z \underbrace{\begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}}_{\vec{v}_3} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}}$$

A vector equation

$$\underbrace{x}_{c_1} \vec{v}_1 + \underbrace{y}_{c_2} \vec{v}_2 + \underbrace{z}_{c_3} \vec{v}_3 = \vec{b}$$

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{b}$$

Can $\vec{b}$ be written as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$? How?

$$\begin{bmatrix} x \\ 0 \\ 4x \end{bmatrix} + \begin{bmatrix} 3y \\ -y \\ 2y \end{bmatrix} + \begin{bmatrix} 4z \\ -z \\ 6z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x + 3y + 4z \\ -y - z \\ 4x + 2y + 6z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x + 3y + 4z &= 3 \\ -y - z &= 1 \\ 4x + 2y + 6z &= 2 \end{aligned}$$

Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 4 & 2 & 6 & 2 \end{array}\right] \overset{-4}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & -10 & -10 & -10 \end{array}\right] \overset{-10}{}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -20 \end{array}\right] \rightarrow 0 = -20, \text{ NOT TRUE}$$

$\Rightarrow$ No solutions!

Inverse matrices

Def (Inverse matrix): Let A be an $\underline{n \times n}$ matrix. (square)

The inverse of A is a matrix A^{-1} such that
($\rightarrow n \times n$)

$$A \cdot A^{-1} = I \quad \underline{\text{and}}$$

$$A^{-1} \cdot A = I$$

Ex:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \; |A| = 4 - 1 = 3 \neq 0$$

$$2^{-1} = \frac{1}{2}$$

$$\rightarrow 2 \cdot 2^{-1} = 2 \cdot \frac{1}{2} = 1$$

$$\rightarrow 2^{-1} \cdot 2 = \frac{1}{2} \cdot 2 = 1$$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \cdot \underbrace{\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\text{some other matrix}}$$

$$= \frac{1}{3}\begin{bmatrix} 4-1 & -2+2 \\ 2-2 & -1+4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2}$$

Can check (DIY):

$$\underbrace{\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\text{the other matrix}} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

Hence, from def: $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

FORMULA (Inverse, $n=2$):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$|A| = ad - bc \neq 0$:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$|A| = ad - bc = 0$:

No inverse of A.

FACTS: i) The inverse of A does not always exist.

A is invertible (i.e., A^{-1} exists) if and only if $|A| \neq 0$.

ii) If A has an inverse, it is unique.