Linear combinations

EBA1180 S26 Lecture 35/11

Ex:

$$\vec{v}_1 = \begin{bmatrix}1\\0\\4\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}3\\-1\\2\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}4\\-1\\6\end{bmatrix}$$

three 3-vectors.

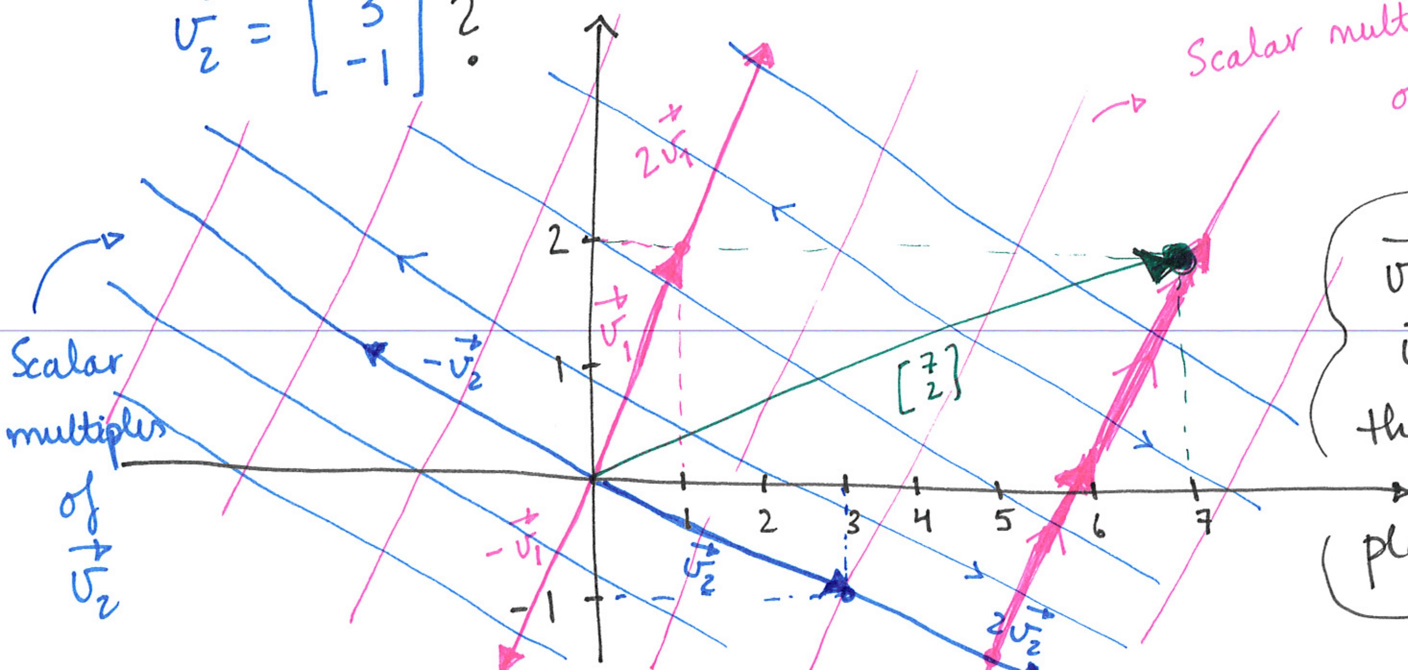
Def. (Linear combination): A linear combination of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ is an expression of the form:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

where c_1, c_2, c_3 are given numbers.

- In general: $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ where $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are m-vectors.

Ex: Is $\begin{bmatrix}7\\2\end{bmatrix}$ a linear combination of $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}3\\-1\end{bmatrix}$?



$\vec{v}_1$ and $\vec{v}_2$ span the whole plane

Want to find c_1 and c_2 s.t:

$$\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2}_{\text{linear combination of } \vec{v}_1 \text{ and } \vec{v}_2} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \longrightarrow \text{A vector equation}$$

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 3c_2 \\ 2c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\left.\begin{aligned} c_1 + 3c_2 &= 7 \\ 2c_1 - c_2 &= 2 \end{aligned}\right\}$$

$x + 3y = 7$
$2x - y = 2$
2×2 linear system

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{-2} \sim \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -7 & -12 \end{array}\right]$$

$$c_1 + 3c_2 = 7$$

$$-7c_2 = -12 \Rightarrow c_2 = \frac{12}{7} \approx \underline{1.72}$$

$$c_1 = 7 - 3c_2 = \ldots = \frac{13}{7} \approx \underline{1.86}$$

Answer: $(c_1, c_2) \approx \underline{\underline{(1.86, 1.72)}}$

What does this mean?

$$\underbrace{1.86 \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v}_1} + \underbrace{1.72 \begin{bmatrix} 3 \\ -1 \end{bmatrix}}_{\vec{v}_2} \approx \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\frac{13}{7} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{12}{7} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Matrix multiplication

Def (Matrix multiplication):

the number of

If A, B matrices and # columns of A = # rows in B,

then

$$\underbrace{A \cdot B}_{\text{matrix multiplication of } A \text{ and } B} \text{ is defined}$$

Ex:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 4 \\ 2 \cdot 3 + 1 \cdot 4 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 11 \\ 10 \end{bmatrix}}}$$

$A \cdot B$

$2 \times \boxed{2} \cdot \boxed{2} \times 1 = 2 \times 1$

columns in A

rows in B

Matrix multiplication is defined

Why is the matrix product defined like this?

Eqn. form: $\begin{cases} x + 2y = 5 \\ 2x + y = 1 \end{cases}$ $\qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$,

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

$$2 \times \cancel{\boxed{2}} \cdot \cancel{\boxed{2}} \times 1 = 2 \times 1$$

SAME

$$A\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \vec{b}$$ Matrix form

START 11.02

Can take powers of square matrices

Ex:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$2 \times \boxed{2} \cdot \cancel{\boxed{2}} \times 2 = 2 \times 2$$

$$= \begin{bmatrix} 1\cdot 1 + 2\cdot 2 & 1\cdot 2 + 2\cdot 1 \\ 2\cdot 1 + 1\cdot 2 & 2\cdot 2 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

<u>FORMULA for $A \cdot B$</u>: If $A \cdot B$ is defined with

$$A = [a_{ij}], \quad B = [b_{ij}], \text{ then } A \cdot B = C = [c_{ij}]$$

where

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

<u>NOTE</u>: $AB \neq BA$

Even though AB is defined, BA may not be.

<u>Ex</u>:

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

$2 \times \cancel{2} \cdot \cancel{2} \times 1 = 2 \times 1 \Rightarrow AB$ is defined.

Same: Matrix prod. is defined

$$\overset{B}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}$$

$2 \times 1 \cdot 2 \times 2 \Rightarrow BA$ is not defined.

<u>NOT SAME</u>: matrix product not defined

Connection to linear systems

Ex:

$$\begin{aligned} x + y + z + w &= 4 \\ x - y \qquad + 2w &= 7 \\ 2x + 3y - z \qquad &= 10 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + y + z + w \\ x - y + 2w \\ 2x + 3y - z \end{bmatrix}$$

$3 \times \cancel{4}$. $\quad \cancel{4} \times 1 = 3 \times 1$

$$= \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} = \vec{b}$$

So the linear system can be written:

$$\boxed{A\vec{x} = \vec{b}}$$

The matrix form of the linear system

Matrix algebra

1) Add, subtract: $A + B$, $A - B$ — A and B are of same size

2) Scalar multiplication: cA, c is a number — Always defined

3) Matrix multiplication: $A \cdot B$ — # columns in A = # rows in B

4) Powers: A^n A square matrix

Ex: $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}}_{2\times 3,\ \text{not square}}^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$

$2\times 3 \cdot 2\times 3$

NOT SAME: Matrix product not defined

Special matrices

The identity matrix: Plays role of 1

$I_2 = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2\times 2}$, $I_3 = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{3\times 3}$, ...,

I_n → Identity matrix of size n ($n\times n$)

We say that $A^0 = I$.

Property: $A \cdot I = A$

Ex: $2\times \cancel{3} \cdot \cancel{3}\times 3 = 2\times 3$

$I \cdot A = A$ for any A, I of suitable dimensions.

Ex: $2\times \cancel{2} \cdot \cancel{2}\times 3 = 2\times 3$

Ex:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 0 & 1\cdot 0 + 2\cdot 1 \\ 2\cdot 1 + 1\cdot 0 & 2\cdot 0 + 1\cdot 1 \end{bmatrix}$$

$$A \cdot I_2$$

$$2\times 2 \cdot 2\times 2 = 2\times 2 \qquad = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A$$

DIY: $I_2 A = A$