EBA 1180
Sect. 4
(28)
526

Partial fractions

Ex: $\int \frac{2}{1-x^2}\,dx$

$$\frac{2}{\underbrace{1-x^2}_{(1+x)(1-x)}} = \frac{A}{1+x} + \frac{B}{1-x}$$

$$0 \cdot x + 2 = (B-A)\,x + (A+B)$$

$$\left.\begin{array}{l} 1)\ 0 = B - A \\ 2)\ 2 = A + B \end{array}\right\} \text{2 linear equations with 2 unknown} \Rightarrow A = B = 1$$

Q: $\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1+x} + \frac{1}{1-x}\,dx$

Partial fractions

$$= \int \frac{1}{1+x}\,dx + \int \frac{1}{1-x}\,dx$$

$$= \frac{1}{1}\ln|1+x| + \frac{1}{(-1)}\ln|1-x| + C$$

SUBSTITUTION:
$u = 1-x$
$du = -dx$
$dx = \frac{1}{-1}\,du$
$= -du$

$$= \ln|1+x| - \ln|1-x| + C$$

$$= \ln\frac{|1+x|}{|1-x|} + C$$

Problem set 27

1) EBA 1180 Spring 17

$$f(x) = 0{,}6 \ln(1+x) + 0{,}4 \ln(1-x), \quad \boxed{0 \le x < 1}$$

Find max-point

a) $$f'(x) = 0{,}6 \frac{1}{1+x} \cdot 1 + 0{,}4 \frac{1}{1-x} \underbrace{(-1)}$$

from chain rule   from chain rule

NB: No issues with division by 0 because of bounds on x

$$= \frac{0{,}6}{1+x} - \frac{0{,}4}{1-x}$$

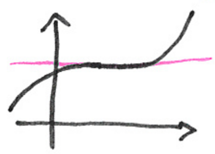
Common denominator

$$= \frac{0{,}6(1-x) - 0{,}4(1+x)}{(1+x)(1-x)}$$

$$= \frac{0{,}6 - 0{,}6x - 0{,}4 - 0{,}4x}{(1+x)(1-x)}$$

$$= \frac{0{,}2 - x}{(1+x)(1-x)}$$

Will be interested in sign of f': Want a factorized form



So: $f'(x) = 0$ gives

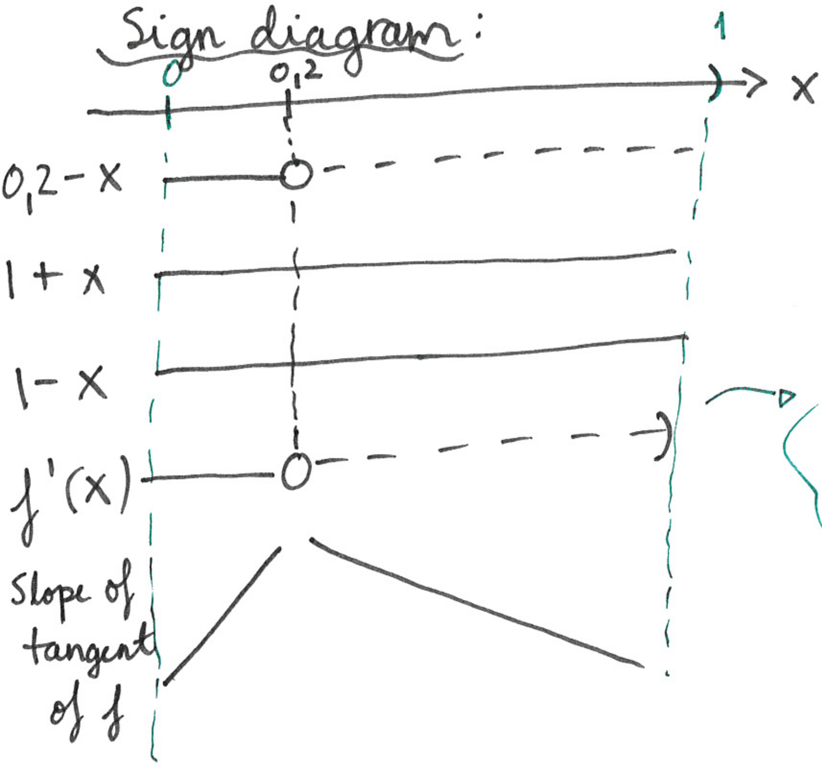
$$\frac{0{,}2 - x}{(1+x)(1-x)} = 0$$

$$0{,}2 - x = 0$$

$$\underline{x = 0{,}2}$$

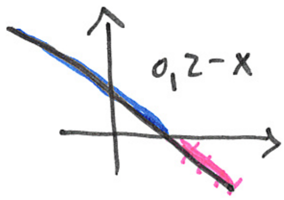
Candidate for a max-point: Need to check whether actually is max. point. To do so: Find sign of derivative

Sign diagram:



$$f'(x) = \frac{0{,}2 - x}{(1+x)(1-x)}$$

Cap diagram between 0 and 1 because of domain



From this, our candidate point $x = 0{,}2$ is in fact a (global) max. point, so

$$\underline{\underline{x^* = 0{,}2}}$$

The max value:

$$f(x^*) = 0{,}6 \ln(1{,}2) + 0{,}4 \ln(0{,}8)$$

Calculator

$$\approx \underline{0{,}0201}$$

approximately equal

b) $$f''(x) = \left(\frac{0{,}2 - x}{1+x} \cdot \frac{1}{1-x} \right)'$$

$$= \frac{(-1)(1+x) - (0{,}2 - x)\cdot 1}{(1+x)^2} \cdot \frac{1}{1-x}$$

Quotient rule combined with product rule

$$+ \frac{0{,}2 - x}{1+x} \frac{1}{(1-x)^2} \underbrace{(-1)}$$

Product rule + chain rule

START 11.01

$$= \frac{(-1 - \cancel{x} - 0{,}2 + \cancel{x})(1-x) + (0{,}2-x)(1+x)}{(1+x)^2(1-x)^2}$$

$-1{,}2$

$$= \frac{-1{,}2 + 1{,}2x + 0{,}2 + 0{,}2x - x - x^2}{(1+x)^2(1-x)^2}$$

$$= \frac{-1 + 0{,}4x - x^2}{(1+x)^2(1-x)^2}$$

Sign of numerator: $-x^2 + 0{,}4x - 1 = 0 \quad | \cdot(-5)$

abc - formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\underset{a}{5}x^2 \underbrace{-2}_{b}x + \underset{c}{5} = 0$$

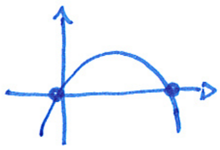
$$x = \frac{2 \pm \sqrt{4 - 4\cdot5\cdot5}}{2\cdot5}$$

$$= \frac{2 \pm \sqrt{4-100}}{10}$$

Negative! No real solutions

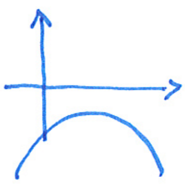
So: $-x^2 + 0{,}4x - 1$ is never 0.

Is it positive or negative?



$x=0 \Rightarrow -0^2 + 0{,}4\cdot 0 - 1 = -1 < 0$

Negative



Numerator is negative.

Denominator sign:

$(1-x)^2 > 0$ and $(1+x)^2 > 0$, hence

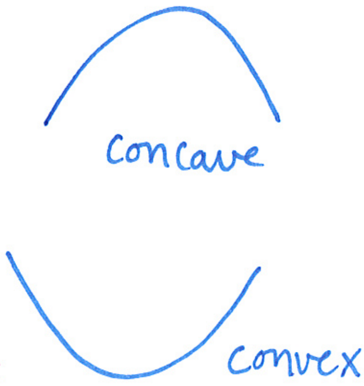
Because of domain: $0 \leq x < 1$

denominator is positive.

$(+ \cdot + = +)$

Hence, $f''(x) < 0$ for all x.

$\frac{-}{+} = -$



Therefore, f is concave.

From a):

c) Show $f(x) < 0$ for $x > \underbrace{2\overbrace{x^*}^{0,2}}_{0,4}$:

From a), $f'(x) < 0$ for $x > x^* = 0,2$.

See sign diagram

Hence, f is decreasing for $x > x^* = 0,2$.

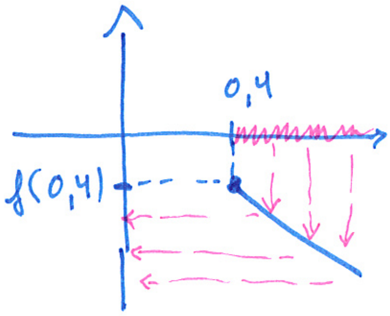
Also:

$$f(2x^*) = f(0,4) = 0,6 \ln(1,4) + 0,4 \ln(0,6)$$

Calculator $\approx -0,0024 < 0$

So: $f(\overbrace{2x^*}^{0,4}) < 0$ and $f(x)$ decreases for all $x > \overbrace{x^*}^{0,2}$, in particular for $x > \underbrace{2x^*}_{0,4}$

Therefore, $f(x) < 0$ when $x > \underbrace{2x^*}_{0,4}$.



d) Sketch graph: We know:

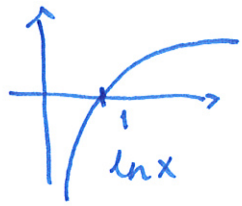
- $f(\underbrace{0,2}_{x^*}) \approx \underbrace{0,0201}_{\text{max. value}}$ From a)
- $f(\underbrace{0,4}_{2x^*}) \approx -0,0024$ From c)

- f is increasing for $x < 0{,}2$ and decreasing for $x > 0{,}2$. → From sign diagram in a)
- f is concave. → From b)
- f is defined on $[0, 1)$ → $[0, 1\rangle$ → From exercise text.

Where does f start?

$$\underline{f(0)} = 0{,}6 \ln(1+0) + 0{,}4 \ln(1-0)$$

$$= 0{,}6 \underbrace{\ln(1)}_{0} + 0{,}4 \underbrace{\ln(1)}_{0} = \underline{0}$$



What happens when we approach 1?

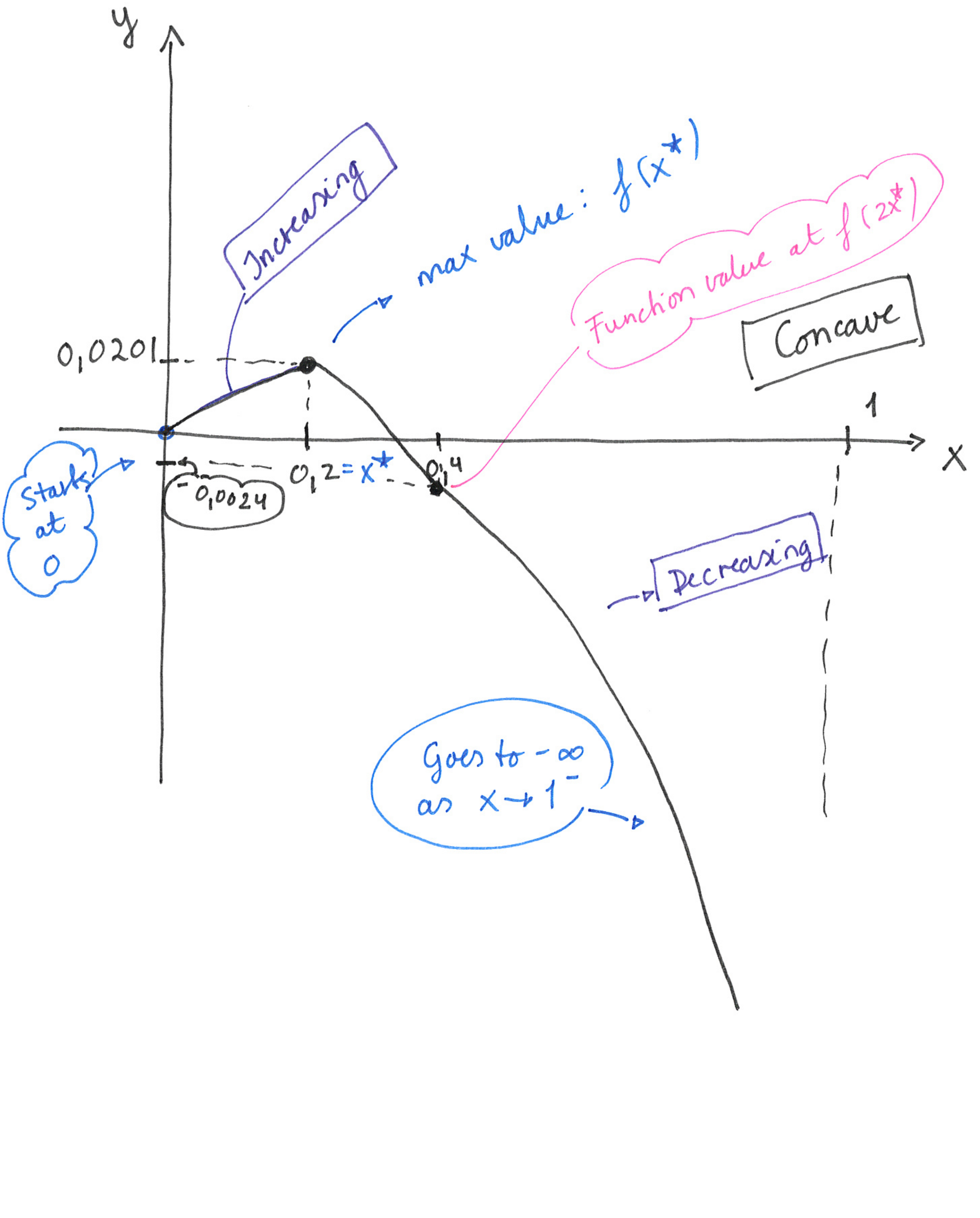
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \underbrace{0{,}6 \ln(\underbrace{1+x}_{\to 2})}_{\to 0{,}6 \ln(2)} + \underbrace{\underbrace{0{,}4 \ln(\underbrace{1-x}_{\to 0})}_{\to -\infty}}_{\to -\infty}$$

From below since $D_f = [0, 1)$

$$= -\infty$$

" $0{,}6 \ln 2 + (-\infty) = -\infty$ "

$\boxed{\text{DRAW ALL OF THIS!}}$



y
x
Increasing
max value : f(x*)
Function value at f(2x*)
Concave
0,0201
1
0,2 = x*
0,4
-0,0024
Starts at 0
Decreasing
Goes to -∞ as x → 1⁻