EBA 1180, lecture 5, 4 Sept. 2024, Runar Ile

Plan
1. Regular cash flows
2. Infinite series and limit values
3. Euler's number and continuous compounding

1. Regular cash flows

A fixed amount is paid every period/term.

Ex Annuity loan (tot. pres. value = what you can borrow)

Ex Saving with a fixed amount each period.
Future value = the balance, what you have saved.

- both give geometric series.

Ex (Term paper 2019a, probl. 6a)
Kåre considers a mortgage with
- monthly payments running for 25 years
- first payment: 5 years from now
- interest: 6 %
- Kåre can pay 15000 each month

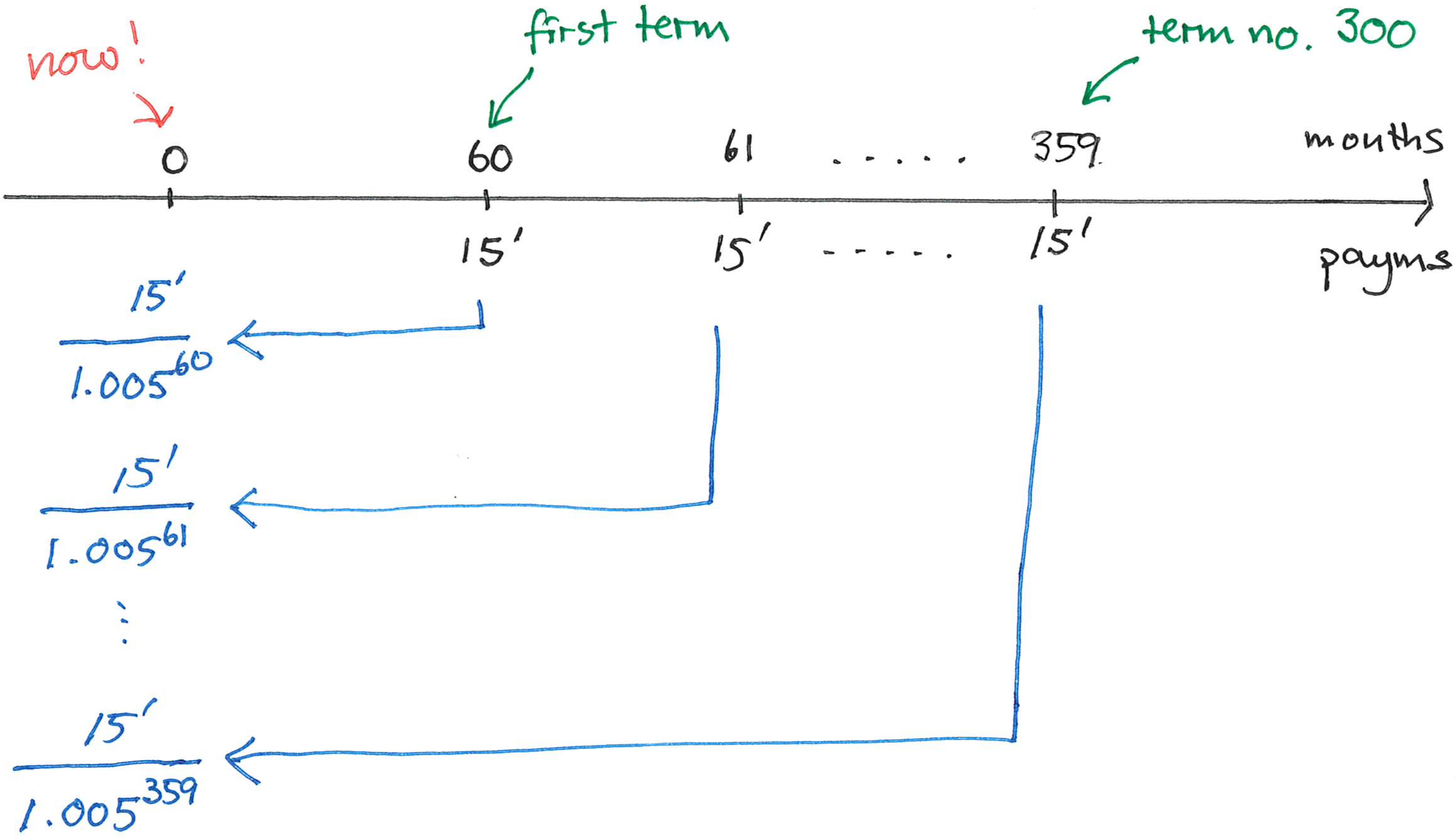
i) Determine the geom. series which gives the tot. pres. value of the cash flow.

ii) Calculate how much Kåre can borrow.

Solution (both i & ii) Kåre can borrow the tot. pres. value of the cash flow.

The period rate is $\frac{6\%}{12} = 0.5\% = 0.005$

Number of periods: $12 \cdot 25 = 300$



The sum (the tot. pres. val.) is a geom. series

with $a_1 = \dfrac{15'}{1.005^{359}}$, $n = 300$, $k = 1.005$

The tot. pres. val. (what Kåre can borrow)

is $\dfrac{15\,000}{1.005^{359}} \cdot \dfrac{1.005^{300} - 1}{0.005} = \underline{\underline{1\,734\,620.76}}$

2. Infinite series and limit values

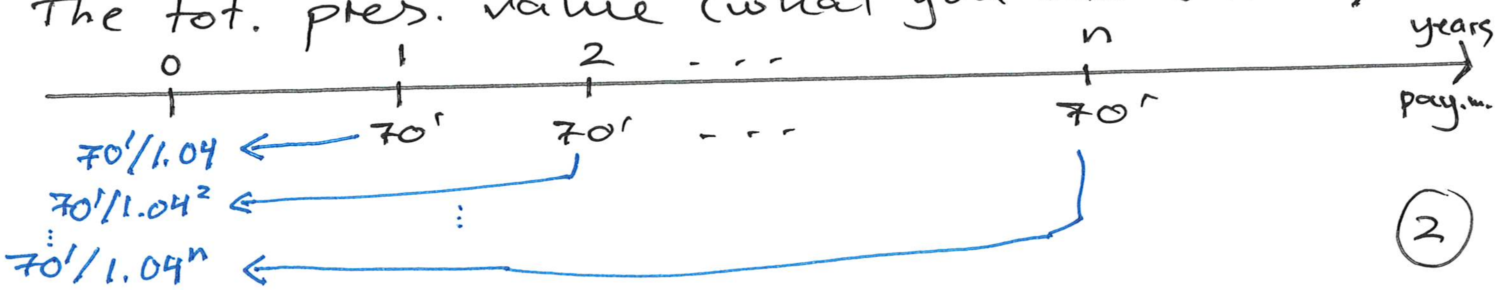
Ex The annuity : 70 000

interest : 4 %

First payment : One year from now

Number of years: n

The tot. pres. value (what you can borrow)



The sum is a geom. series with

$a_1 = \frac{70'}{1.04^n}$, n terms, $k = 1.04$. The sum is then

$$\frac{70'}{1.04^n} \cdot \frac{1.04^n - 1}{0.04} = \frac{70' \cdot (1.04^n - 1) : 1.04^n}{0.04 \cdot 1.04^n : 1.04^n}$$

$$= \frac{70' \cdot \left(1 - \frac{1}{1.04^n}\right)}{0.04} \xrightarrow[n \to \infty]{} \frac{70'}{0.04}$$

approaches 0 when $n \to \infty$

" n goes to infinity"

= " n becomes bigger and bigger without bounds"

meaning: the tot. pres. val. is approaching $\frac{70'}{0.04}$ when n grows without bounds.

$$= \underline{\underline{1\,750\,000}}$$

<u>Conclusion</u> If you pay the bank 70000 each year, starting next year, and the interest is 4% and you pay <u>forever</u>, then you can borrow 1.75 mill.

start: 11.01

3. Euler's number and continuous compounding

Ex You deposit 1000 into an account with 12% nominal interest.

compounding	balance after 1 year
Annual	$1000 \cdot 1.12 = 1120.00$
Half year	$1000 \cdot 1.06^2 = 1123.60$
Quaterly	$1000 \cdot 1.03^4 = 1125.51$
Monthly	$1000 \cdot 1.01^{12} = 1126.83$
Daily	$1000 \cdot \left(1 + \frac{12\%}{365}\right)^{365} = 1127.47$

Pattern (n periods) $1000 \cdot \left(1 + \frac{0.12}{n}\right)^n$

Euler's number: $e = 2.718281\ldots$

Calculator: 1 $\boxed{e^x}$

Calculate: $1000 \cdot e^{0.12} = 1127.50$

1000 $\boxed{\times}$ 0.12 $\boxed{e^x}$ $\boxed{=}$

Euler's number e is defined as the limit of $\left(1 + \frac{1}{n}\right)^n$ when $n \to \infty$

Write: $\left(1 + \frac{1}{n}\right)^n \xrightarrow[n \to \infty]{} e$

Ex $\left(1 + \frac{1}{1000}\right)^{1000} = 2.71692\ldots$ $\left(1 + \frac{1}{1\text{mill.}}\right)^{1\text{ mill}} = 2.718280\ldots$

Back to the example with 12 %.

$$\left(1+\frac{0.12}{n}\right)^n = \left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^n$$

$$= \Bigg[\underbrace{\left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^{\frac{n}{0.12}}}_{\text{approaches } e \text{ when } n\to\infty}\Bigg]^{0.12} \xrightarrow[n\to\infty]{} e^{0.12}$$

so $1000\cdot\left(1+\frac{0.12}{n}\right)^n \xrightarrow[n\to\infty]{} 1000\cdot e^{0.12}$

After 1 year with 12 % nominal interest and <u>continuous compounding</u> the deposit of 1000 has increased to $1000\cdot e^{0.12} = 1127.50$

the growth factor for 1 year with continuous compounding

Annual growth factor : $e^{0.12} = 1.12750$

The <u>effective</u> interest : $e^{0.12}-1 = 0.12750 = 12{,}750\%$

After 2 years of continuous compounding :

$1000\cdot e^{0.12}\cdot e^{0.12} = 1000\cdot e^{0.12+0.12} = 1000\cdot e^{0.24} = \underline{1271.25}$

After 1 month : $1000\cdot\left(e^{0.12}\right)^{\frac{1}{12}} = 1000\cdot e^{0.01} = \underline{\underline{1010.05}}$