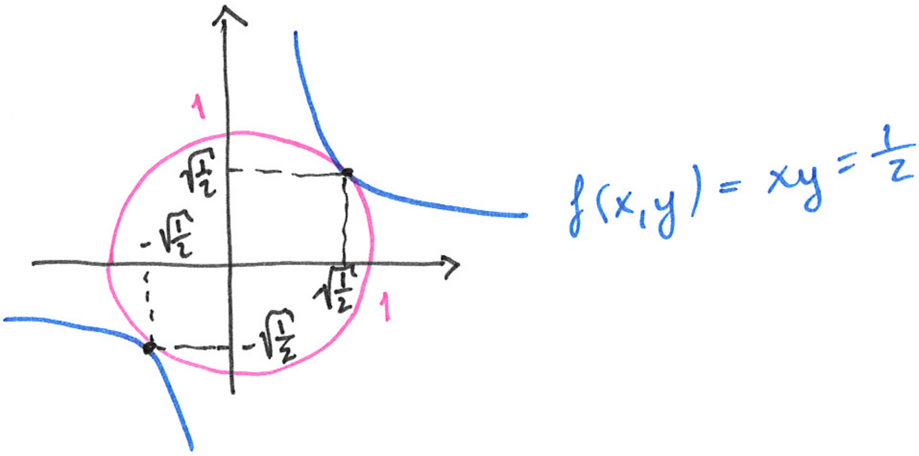
Interpretation of Lagrange multipliers

EBA 1180
Lect. 47
Spring 25

Ex: max/min $f(x,y) = xy$ when $\underbrace{x^2+y^2}_{g(x,y)} = 1$ → Circle with center $(0,0)$, $r=1$

$f_{max} = \frac{1}{2}$ at

$\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$, $\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$ with $\lambda = \frac{1}{2}$



Def: Consider max $f(x,y) = xy$ with $x^2+y^2 = a$ → Radius of circle: $\sqrt{a}$ → parameter

max. point: $(x^*(a), y^*(a))$ → since solution depends on a

Max. value: $f(x^*(a), y^*(a)) = f^*(a)$

Ex: $a=1$: $x^*(1) = \sqrt{\frac{1}{2}}$, $y^*(1) = \sqrt{\frac{1}{2}}$, $f^*(1) = \frac{1}{2}$

OR: $x^*(1) = -\sqrt{\frac{1}{2}}$, $y^*(1) = -\sqrt{\frac{1}{2}}$, $f^*(1) = \frac{1}{2}$

$$\boxed{\text{Result: } \underbrace{\lambda}_{\text{Lagrange multiplier}} = \frac{d f^*(a)}{d a}}$$

$d f^*(a)$ → Change in the max value when a changes

$d a$ → small change in a

Interpretation of λ: λ is the marginal change in the max (min) value per unit change in the constant a in the constraint $g(x,y)=a$.

Ex: USE THIS TO APPROXIMATE OPTIMAL VALUES:

$a=2$: $$f^*(2) \approx f^*(1) + \Delta a \underbrace{\frac{d f^*(a)}{d a}}_{=\lambda \text{ from Result}}$$

Def. of derivative:

$$f'(a) \approx \frac{\Delta f(a)}{\Delta a} = \frac{f(a+h)-f(a)}{h}$$

Actually: $$f'(a) = \lim_{\Delta a \to 0} \frac{\Delta f(a)}{\Delta a} = \lim_{h \to 0} \ldots$$

$$\to \Delta a \, f'(a) \approx \Delta f(a)$$

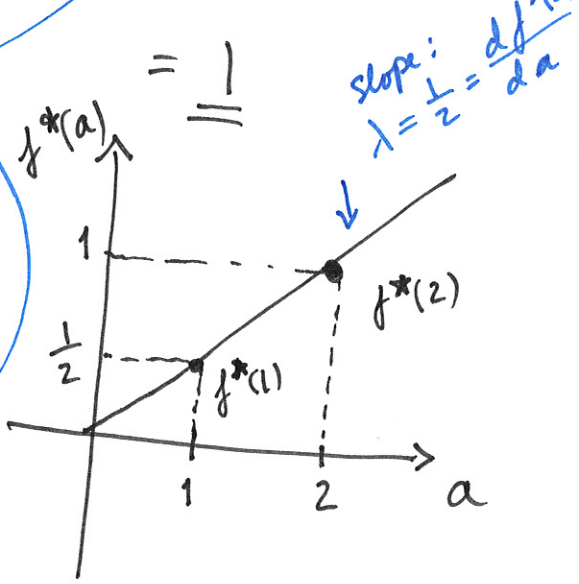
$$\Rightarrow \Delta f^*(a) \approx \Delta a \, (f^*)'(a)$$

$$f^*(2) - f^*(1) \approx (2-1) \left.\frac{d f^*(a)}{d a}\right|_1$$

$$= \frac{1}{2} + 1 \cdot \lambda \quad \left(\lambda = \frac{1}{2}\right)$$

$$= \frac{1}{2} + 1 \cdot \frac{1}{2}$$

$$= \underline{\underline{1}}$$



Ex. ctd:

$$\max f(x,y) = xy \quad \text{when} \quad \underbrace{x^2+y^2=a}_{\text{Circle, center } (0,0),\ r=\sqrt{a},\ a>0}$$

EVT:
- $f(x,y)$ continuous? Yes.
- Compact constraint set?
  - → Closed? Yes. (=)
  - → Bounded? Yes. (circe)

$\Rightarrow$ EVT holds! The problem has a max (and a min.)!

Type ii) points: Admissible points with degenerate constraint?

$$g(x,y) = x^2 + y^2$$

$g'_x = 2x = 0 \Rightarrow x = 0$

$g'_y = 2y = 0 \Rightarrow y = 0$

Then:

$x^2 + y^2 = 0^2 + 0^2 = 0 \neq \underbrace{a}_{a>0}$

No admissible points with degenerate constraint.

Type i) points: $\mathcal{L}(x,y) = xy - \lambda(x^2 + y^2 - a)$ ← $f - \lambda(g - a)$

FOC:

(1): $\mathcal{L}'_x = y - 2x\lambda = 0 \Rightarrow \boxed{y = 2\lambda x}$

(2): $\mathcal{L}'_y = x - 2y\lambda = 0 \Rightarrow x - 2 \cdot 2\lambda x \cdot \lambda = 0$

C: $x^2 + y^2 = a$

$x - 4\lambda^2 x = 0$

$x(1 - 4\lambda^2) = 0$

3 cases:

$x = 0$:

From (1): $y = 2\lambda \cdot 0 = 0$

From C: $0^2 + 0^2 = a$

$0 = a$

For $a = 0$: $(0, 0; \lambda)$ λ can be anything, $f = 0$

If $a \neq 0$: No candidate pt.

$\lambda^2 = \frac{1}{4}$:

$\lambda = \frac{1}{2}$:

From (1): $y = 2 \cdot \frac{1}{2} x = x$

From C: $y^2 + y^2 = a$

$2y^2 = a$

$x = y = \pm\sqrt{\frac{a}{2}}$

Candidates: $\left(\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}; \frac{1}{2}\right)$, $f = \frac{a}{2}$

and $\left(-\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}; \frac{1}{2}\right)$, $f = \frac{a}{2}$

$\lambda = -\frac{1}{2}$:

From (1): $y = 2(-\frac{1}{2})x = -\frac{x}{2}$

From C: $y^2 + y^2 = a$

$2y^2 = a$

$y = \pm\sqrt{\frac{a}{2}}$

Candidates: $\left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}; -\frac{1}{2}\right)$, $f = \frac{a}{2}$

and $\left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}; -\frac{1}{2}\right)$, $f = -\frac{a}{2}$

Conclusion: $f_{max} = \frac{a}{2}$ at the max. points

$\left(\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right)$ and $\left(-\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right)$

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$$f^*(a) = \frac{a}{2} = \underbrace{\frac{1}{2}}_{\lambda} a$$

$$\frac{d f^*(a)}{d a} = \frac{1}{2} = \lambda$$

Ref. result!

Kuhn-Tucker problems

→ Optimization problems with closed inequality constraints ($\leq$, $\geq$)

Ex:

$$\boxed{\max \quad f(x,y) = x^2 y^2 \quad \text{when} \quad x^2 + y^2 + x^2 y^2 \leq 3}$$

Kuhn-Tucker problem

Candidate points

(1) Boundary points: Boundary ⇒ = constraint

⇒ $\boxed{\text{Lagrange problem}}$ ⇒ Solve via standard Lagrange technique.

(2) Interior points: Stationary points or other interior critical points for f.

Ex: $f'_x = 2xy^2 = 0 \Rightarrow x=0$ or $y=0$

$f'_y = 2x^2y = 0 \Rightarrow x=0$ or $y=0$

Candidates: • $x=0$: INTERIOR POINT:

$(0,y)$ when $0^2+y^2+0^2y^2 < 3$

$$y^2 < 3$$

$$-\sqrt{3} < y < \sqrt{3}$$

• $y=0$: INTERIOR POINT:

$(x,0)$ when $x^2+0^2+x^2 0^2 < 3$

$$x^2 < 3$$

$$-\sqrt{3} < x < \sqrt{3}$$

The boundary: Lagrange problem

boundary: $=$

max $f(x,y) = x^2y^2$ when $x^2+y^2+x^2y^2 = 3$

Type i): $\mathcal{L}(x,y;\lambda) = x^2y^2 - \lambda(x^2+y^2+x^2y^2-3)$

FOC: $\mathcal{L}'_x = 2xy^2 - \lambda(2x + 2xy^2) = 0$ (1):

$\mathcal{L}'_y = 2x^2y - \lambda(2y + 2yx^2) = 0$ (2):

C: $x^2+y^2+x^2y^2 = 3$ (3):

From (1): $2x(y^2 - \lambda - \lambda y^2) = 0$ → $x = 0$ OR $y^2 - \lambda - \lambda y^2 = 0$

From (2): $2y(x^2 - \lambda - \lambda x^2) = 0$ → $y = 0$ OR $x^2 - \lambda - \lambda x^2 = 0$

Check all combinations:

a) $x=0,\ y=0$: From (3): $0^2 + 0^2 + 0^2 0^2 = 3$

$$0 = 3$$

NOT TRUE ⇒ NO candidates.

b) $x=0,\ x^2 - \lambda - \lambda x^2 = 0$: → $0^2 - \lambda - \lambda \cdot 0^2 = 0 \Rightarrow -\lambda = 0 \Rightarrow \lambda = 0$

From (3): $0^2 + y^2 + 0^2 y^2 = 3 \iff y^2 = 3$

$$y = \pm\sqrt{3}$$

$f=0$

Candidates: $(0, \sqrt{3}; 0)$, $(0, -\sqrt{3}; 0)$

$f=0$

Symmetry (or online!)

c) $y=0,\ y^2 - \lambda - \lambda y^2 = 0$: Candidates:

$f=0$

$\cancel{(0,\sqrt{3};0)}$ $(\sqrt{3}, 0; 0)$, $(-\sqrt{3}, 0; 0)$

$f=0$

d) $y^2 - \lambda - \lambda y^2 = 0,\ x^2 - \lambda - \lambda x^2 = 0$: Online.

Candidates: $(1, 1; \frac{1}{2})$, $(1, -1; \frac{1}{2})$, $(-1, 1; \frac{1}{2})$, $(-1, -1; \frac{1}{2})$

For all: $\boxed{f = 1}$

Type ii) : Admissible pts. with degenerate constraint:

None : Online.

Conclude : $f_{max} = 1$ at $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$.

Need arg. for why a max. exists: See online !

EVT: Need to show D is bounded