(Look at Lagrange problem part of the Kuhn Tucker problem)

Ex: $\max f(x,y) = x^2 y^2$ when $x^2+y^2+x^2y^2 = 3$

$\mathcal{L} = x^2y^2 - \lambda(x^2+y^2+x^2y^2-3)$

CANDIDATE POINTS:

2 kinds
i) Stationary pts. of $\mathcal{L}$
ii) Degenerate constraint
(From Theorem)

i) Stationary points of $\mathcal{L}$:

$$\mathcal{L}'_x = 2xy^2 - \lambda(2x + 2xy^2) = 0 \quad (1)$$

$$\mathcal{L}'_y = x^2 \cdot 2y - \lambda(2y + x^2 2y) = 0 \quad (2)$$

C: $\mathcal{L}'_\lambda = x^2+y^2+x^2y^2 = 3 \quad (3)$

From (1):

$$2x(y^2 - \lambda - \lambda y^2) = 0 \rightarrow x = 0 \text{ OR } y^2-\lambda-\lambda y^2 = 0$$

From (2):

$$2y(x^2 - \lambda - \lambda x^2) = 0 \rightarrow y = 0 \text{ OR } x^2-\lambda-\lambda x^2 = 0$$

(Check all combinations:) ⇓

a) $x=0,\ y=0$: From (3): $0^2+0^2+0^20^2 = 3$; Impossible!
$\Rightarrow$ No candidates.

b) $x=0,\ x^2-\lambda-\lambda x^2 = 0$: $\rightarrow$ $0^2-\lambda-0 = 0 \Rightarrow \lambda = 0$

From (3): $0^2+y^2+\underbrace{0^2y^2}_{0} = 3 \Rightarrow y = \pm\sqrt{3}$

So: Candidates: $(0, \sqrt{3};\ \underbrace{0}_{\lambda})$, $(0, -\sqrt{3};\ \underbrace{0}_{\lambda})$

$f(0,\sqrt{3}) = 0$ $\qquad$ $f(0,-\sqrt{3}) = 0$

(c) $y=0,\ y^2-\lambda-\lambda y^2=0$:

$$0^2-\lambda-\lambda\cdot 0^2=0 \Rightarrow \lambda=0$$

From (3):

$$x^2+0^2+x^2 0^2=3 \Rightarrow x^2=3, \text{ so } x=\pm\sqrt{3}$$

Candidates: ~~$(0,\sqrt{3},0),(0,-\sqrt{3},0)$~~

$(\sqrt{3},0;0),\ (-\sqrt{3},0;0)$

d) $y^2-\lambda-\lambda y^2=0,\ x^2-\lambda-\lambda x^2=0$:

$$y^2=\lambda(1+y^2) \Rightarrow \lambda=\frac{y^2}{1+y^2}$$ never 0

$$x^2=\lambda(1+x^2) \Rightarrow \lambda=\frac{x^2}{1+x^2}$$ (★) never 0

$\lambda=\lambda$ $\Rightarrow$ $$\frac{y^2}{1+y^2}=\frac{x^2}{1+x^2} \quad \Big| \cdot(1+y^2)(1+x^2)$$

$$y^2(1+x^2)=x^2(1+y^2)$$

$$y^2+\cancel{y^2x^2}=x^2+\cancel{x^2y^2}$$

$$y^2=x^2$$

$$y=\pm x : (★)$$

From (3):

$$x^2 + x^2 + x^2 x^2 = 3$$

$$x^4 + 2x^2 - 3 = 0 \quad , \quad \text{let } u = x^2:$$

$u^2 + 2u - 3 = 0$; a quadratic eqn:

$$u = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{16}}{2} = \frac{-2 \pm 4}{2} = \begin{cases} 1 \\ -3 \end{cases}$$

So $x^2 = 1$ or $\cancel{x^2 = -3}$

NOT POSSIBLE!

$$x = \pm 1$$

Hence, $y = \pm x = \pm 1$ → OBS! 4 combinations:

From (*)

$x = 1$: $y = 1$, $y = -1$

$x = -1$: $y = -1$, $y = 1$

Also, $\lambda = \dfrac{x^2}{1 + x^2} = \dfrac{1}{2} \Rightarrow$

From (*)

Candidates: $(1, 1; \frac{1}{2}), (1, -1; \frac{1}{2}),$
$(-1, 1; \frac{1}{2}), (-1, -1; \frac{1}{2})$ (★)

For all of these: $f = 1$

$x^2 + y^2$ (sign irrelevant due to square)

ii) Admissible points with degenerate constraints:

$$g(x,y) = x^2 + y^2 + x^2y^2 = 3$$

$$g'_x = 2x + 2xy^2 = 0 \Rightarrow 2x\underbrace{(1+y^2)}_{\text{Never } 0} = 0 \Rightarrow x = 0$$

$$g'_y = 2y + x^2 2y = 0 \Rightarrow 2y\underbrace{(1+x^2)}_{\text{Never } 0} = 0 \Rightarrow y = 0$$

$\Rightarrow (x,y) = (0,0)$; But this is not admissible since then

$$g(0,0) = 0^2 + 0^2 + 0^2 0^2 = 0 \neq 3,$$ so the constraint doesn't hold.

Hence, no candidates of type ii)

Conclusion: $f_{max} = 1$ at $(1,1), (1,-1), (-1,1)$ and $(-1,-1)$ with $\lambda = \frac{1}{2}$

Is there a max?

D: $x^2+y^2+x^2y^2 = 3$

Closed? ✓

Bounded? ✓

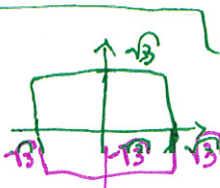
Note that WHY? x^2, y^2 and $x^2y^2 \geq 0 \ \forall x, y$. Hence, both x^2 and y^2 must be ≤ 3. It then:

$x^2 \leq 3$ and $y^2 \leq 3 \iff$

$\sqrt{3} \leq x \leq \sqrt{3}$

$\sqrt{3} \leq y \leq \sqrt{3}$

admissible x's and y's fit into this ~~rectangle~~:



So D is bounded ⇒ D is compact & f cont. ⇒ EVT holds ⇒ ∃ max and min.