More Lagrange problems

Thm: If (x^*, y^*) is a max/min in a Lagrange problem:

$$\boxed{\max/\min\ f(x,y) \text{ where } g(x,y) = a}$$

Then either

i) There is a λ s.t. $(x^*, y^*; \lambda)$ satisfies the Lagrange constraints FOC + C:

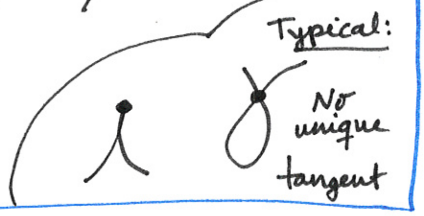
$$\text{FOC} \begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases} \quad \text{and} \quad \underline{C}:\ g(x,y) = a$$

Lagrangian: $\mathcal{L}(x,y) = f(x,y) - \lambda\,(g(x,y) - a)$

OR:

ii) The constraint is degenerate at (x^*, y^*), i.e.:

$$g'_x = 0 \quad \text{and} \quad g(x,y) = a$$
$$\text{and} \quad g'_y = 0$$

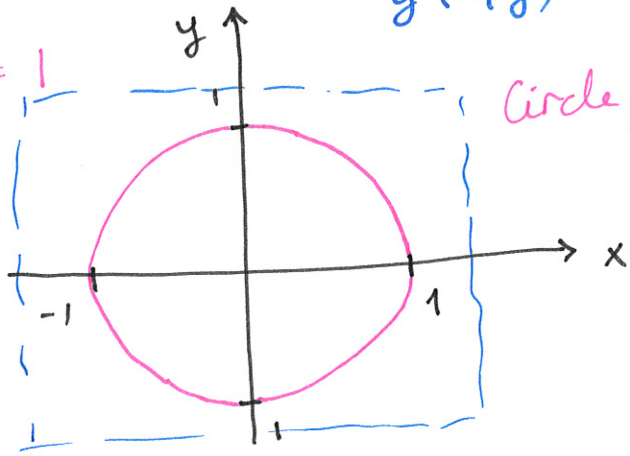


Ex: $\max/\min\ f(x,y) = xy$ when $\underbrace{x^2 + y^2}_{g(x,y)} = 1$

NOTE: $D:\ x^2 + y^2 = 1$ is compact:

Closed ✓ (=)

Bounded ✓



Circle, center $(0,0)$, $r = \sqrt{1} = 1$

f is continuous

$\Downarrow$

EVT: f has a max and ~~a~~ min over D.

→ Degenerate constraint? $g'_x = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

$g'_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$,

but then: $x^2 + y^2 = 0^2 + 0^2 = 0 \neq 1$, so the constraint doesn't hold.

Hence, there are no admissible points with degenerate constraint ⇒ No type ii) candidates (ref. Thm.)

NB: Holds in general for circles.

Type (i) candidates?

Lagrangian: $\mathcal{L}(x,y) = xy - \lambda\,(x^2 + y^2 - 1)$

$$\boxed{\begin{aligned} \text{FOC:}\quad & \mathcal{L}'_x = y - \lambda \cdot 2x = 0 \\ & \mathcal{L}'_y = x - \lambda \cdot 2y = 0 \\ \text{C:}\quad & x^2 + y^2 = 1 \end{aligned}}$$

3 equations with 3 unknowns: x, y, λ

$y = 2\lambda x$

$x - \lambda \cdot 2(2\lambda x) = 0$

$x \cdot (1 - 4\lambda^2) = 0$

$x = 0$

$y = 2\lambda x = 2\lambda \cdot 0 = 0$

$x^2 + y^2 = 0^2 + 0^2 = 0 \neq 1$

so the constraint doesn't hold

⇒ Not a candidate point

$1 - 4\lambda^2 = 0$

$\lambda^2 = \frac{1}{4}$

$\lambda = \frac{1}{2}$:

$\lambda = -\frac{1}{2}$:

$\lambda = \frac{1}{2}$:

$y = 2 \cdot \frac{1}{2} x = x$

C: $x^2 + y^2 = 1$

$x^2 + x^2 = 1$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}} = y$

Candidate points:

$\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$, $\to$ $f(x,y) = xy = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}} = \frac{1}{2}$

$\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$ But with $-$ times $-$ $= +$

$\boxed{\text{MAX}}$

$\lambda = -\frac{1}{2}$:

$y = 2 \cdot \left(-\frac{1}{2}\right) x = -x$

C: $x^2 + y^2 = 1$

$x^2 + (-x)^2 = 1$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$

$y = -x$

Candidate points:

$\left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right) \to$ $f(x,y) = xy = \sqrt{\frac{1}{2}}\left(-\sqrt{\frac{1}{2}}\right) = -\frac{1}{2}$

$\left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$ SAME calc., opposite order

$\boxed{\text{MIN}}$

o—o

Ex: max/min $f(x,y) = x^2 + y^2$ when $\overbrace{xy = 1}^{D}$

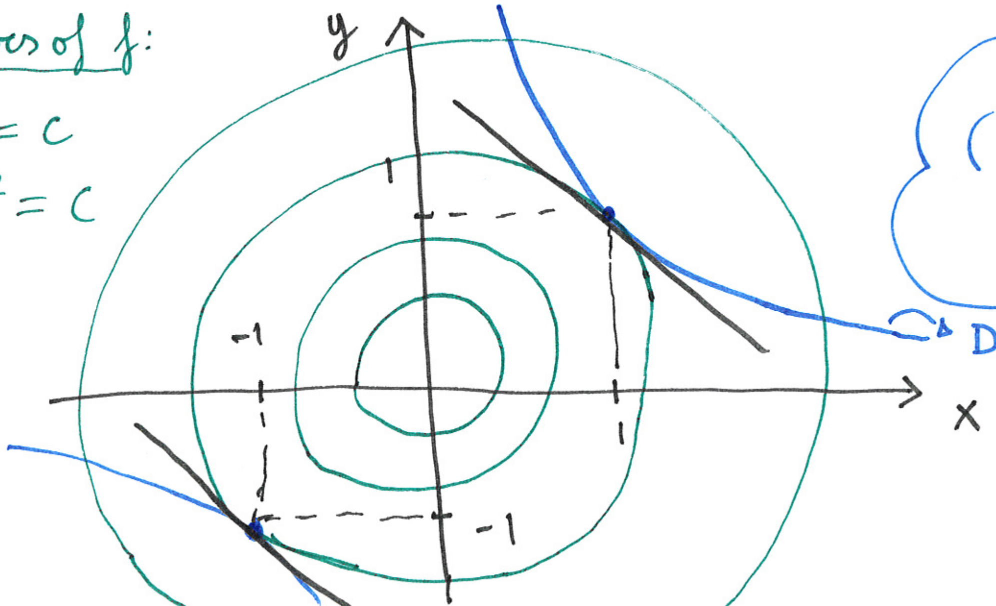
Level curves of f:

$f(x,y) = c$

$x^2 + y^2 = c$

$y = \frac{1}{x}$

(x can't be 0 since then $xy = 0 \neq 1$)



③

START : 11.03

EVT? Closed? ✓
D: Bounded? NO! Can't be boxed in
f continuous? ✓

→ Can't use EVT

Type i) candidates: $\mathcal{L}(x,y) = x^2 + y^2 - \lambda(xy-1)$

FOC:
$$\mathcal{L}'_x = 2x - \lambda y = 0$$
$$\mathcal{L}'_y = 2y - \lambda x = 0$$

C:
$$xy = 1$$

3 eqns, 3 unknowns
x, y, λ

$$2x = \lambda y$$
$$x = \frac{\lambda y}{2}$$

$$2y - \lambda\left(\frac{\lambda y}{2}\right) = 0 \quad \Big| \cdot 2$$

$$4y - \lambda^2 y = 0$$

$$y(4 - \lambda^2) = 0$$

3 cases:

$y = 0$:
$$x = \frac{\lambda \cdot 0}{2} = 0$$
C:
$$xy = 0 \cdot 0 = 0 \neq 1$$
Not a candidate pt. because constraint doesn't hold.

$\lambda = 2$
$$x = \frac{2y}{2} = y$$
C: $xy = 1$
$$x^2 = 1$$
$$x = \pm 1$$
Candidate pts:
$(1, 1)$ and $(-1,-1)$

$f = x^2 + y^2 = 1^2 + 1^2 = 2$

$f = 2$

$\lambda = -2$
$$x = \frac{-2y}{2} = -y$$
C: $xy = 1$
$$-y^2 = 1$$
$$\underbrace{y^2 = -1}_{\text{NOT POSSIBLE!}}$$
⇓
No candidate points

Type ii) candidates : Admissible points with degenerate constraint ?

$\boxed{g(x,y) = xy}$

$$\left.\begin{array}{l} g'_x = y = 0 \\ g'_y = x = 0 \end{array}\right. \Rightarrow \quad xy = 0 \cdot 0 = 0 \neq 1$$

$\Rightarrow$ There are no admissible points with degenerate constraint.

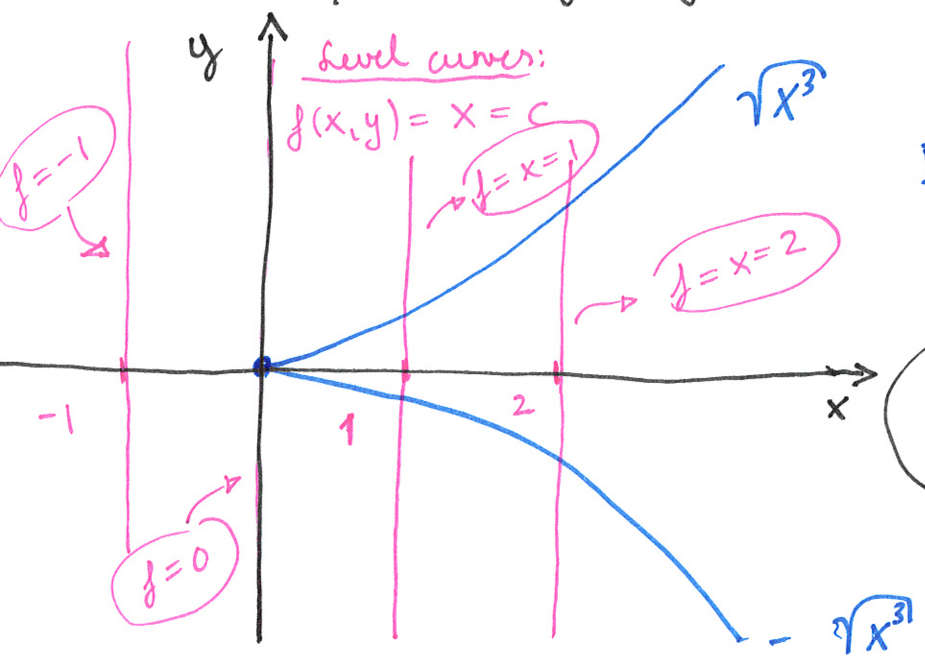
CONCLUSION : $f_{min} = 2$ at $(1,1)$ and $(-1,-1)$ with $\lambda = 2$.

No maximum since $y = \frac{1}{x}$ will satisfy the constraint. Can let $x \to \infty$. Then, $y \to 0$, but is admissible. Then,

$$f(x,y) = x^2 + y^2 \longrightarrow \text{“} \infty^2 + 0^2 \text{”} = \infty$$

Ex : max/min $f(x,y) = x$ when

$$\underbrace{y^2 - x^3 = 0}_{}$$



D: $y^2 = x^3$

$y = \pm\sqrt{x^3}$

NB: Only def. for $x^3 \geq 0 \Rightarrow x \geq 0$

D unbounded: Can't use EVT ?

Type ii) candidates: Admissible pts. with degenerate constraints?

$g'_x = -3x^2 = 0 \Rightarrow x = 0$

$g'_y = 2y = 0 \Rightarrow y = 0$, but then

$g(x,y) = g(0,0) = 0^2 - 0^3 = 0$, so $(0,0)$ is on D. Hence, $(0,0)$ is an admissible pt. with degenerate constraint.

Type i) candidates: $\mathcal{L}(x,y) = x - \lambda(y^2 - x^3)$

FOC:

$$\mathcal{L}'_x = 1 + \lambda \cdot 3x^2 = 0 \quad :(1)$$

$$\mathcal{L}'_y = -\lambda \cdot 2y = 0 \quad :(2)$$

C:

$$y^2 - x^3 = 0 \quad :(3)$$

(2): $-\lambda \cdot 2y = 0$

$\lambda = 0$:

(1): $1 + 0 \cdot 3x^2 = 0$

$1 = 0$

NOT TRUE! No candidate pt.

$y = 0$:

(3): $0^2 - x^3 = 0 \Rightarrow x = 0$

(1): $1 + \lambda \cdot 3 \cdot 0^2 = 0$

$1 = 0$

NOT TRUE! No candidate pt.

Hence, we have no ordinary candidate points.

All we have: Admissible point with degenerate constraint; $(0, 0)$

From the Figure: We see that this is the minimum point. No maximum (from figure, OR no candidates)

D intersects level curves with larger and larger value

OR: $y = \pm\sqrt{x^3}$ makes constraint hold. Can choose arbitrarily large $x \Rightarrow f(x,y) = x \longrightarrow \infty$

$\Rightarrow$ NO MAX.