Examples: Optimization

EBA1180
Sect. 4.4
Spring 25

Ex: $f(x,y) = x^2y^3 + y^2 - 2y$,

$D_f = \mathbb{R}^2$ → No boundary: An unconstrained optimization problem

Q: What's the plan to: max/min $f(x,y)$?

In general:

Candidate points:

i) Stationary points

ii) Other critical points: Partial derivative not def.

iii) Boundary points: None here.

CLASSIFY CANDIDATES: 2nd derivative test

i) Stationary points:

$$\begin{cases} f'_x = 2xy^3 = 0 \Rightarrow x=0 \text{ or } y=0 \\ f'_y = 3x^2y^2 + 2y - 2 = 0 \end{cases}$$

Can both x and y be 0?

$3\cdot 0^2 \cdot 0^2 + 2\cdot 0 - 2 = -2 \neq 0$

$\Rightarrow (0,0)$ is not possible!

2 cases

$x=0$:	$y=0$:
$3\cdot 0^2 \cdot y^2 + 2y - 2 = 0$	$3x^2\cdot 0 + 2\cdot 0 - 2 = 0$
$2y = 2$	$-2 = 0$
$y = 1$	Not true! $\Rightarrow$ $(x,0)$ not possible.

Hence, only one stationary point $(x^*, y^*) = (0, 1)$.

Note: $f(0,1) = 0^2\cdot 1^3 + 1^2 - 2\cdot 1 = \underline{\underline{-1}}$

ii) Other critical points: Partial derivatives are defined everywhere $\Rightarrow$ No other critical points.

iii) Boundary: No boundary.

$\Rightarrow$ The stationary point is the only candidate point.

Classification of stationary point

f''_{xx} $\qquad\qquad$ f''_{xy}

$$H(f)(x,y) = \begin{bmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{bmatrix}$$

f''_{yx} $\qquad\qquad$ f''_{yy}

Insert $(0,1)$: $\quad H(f)(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, so

$\det H(f)(0,1) = 2 \cdot 2 - 0 \cdot 0 = 4 > 0$,

$\operatorname{tr} H(f)(0,1) = 2 + 2 = 4 > 0$

$\Downarrow$ 2nd derivative test

$f(0,1) = -1$ is a local minimum.

NOTE: If $x = 1$, then:

$$f(1,y) = \underbrace{y^3 + y^2 - 2y}$$

the y^3 will overpower $y^2 - 2y$

Can be made arbitrarily small. Test: Calculator →

$y = -100, -1000, -10\,000$.

The $y^3 + y^2$ will overpower $-2y$

OR be made arbitrarily large

(try: $y = 100, 1000, 10\,000$)

$$\Downarrow$$

f has no (global) max or (global) min.

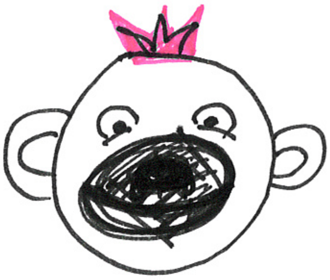
NB: Example of a function with only one stationary pt., local min, but still no global minimum.

Ex. sheet 43

*(?)

5) d) max/min $f(x,y) = xy(x^2 - y^2)$

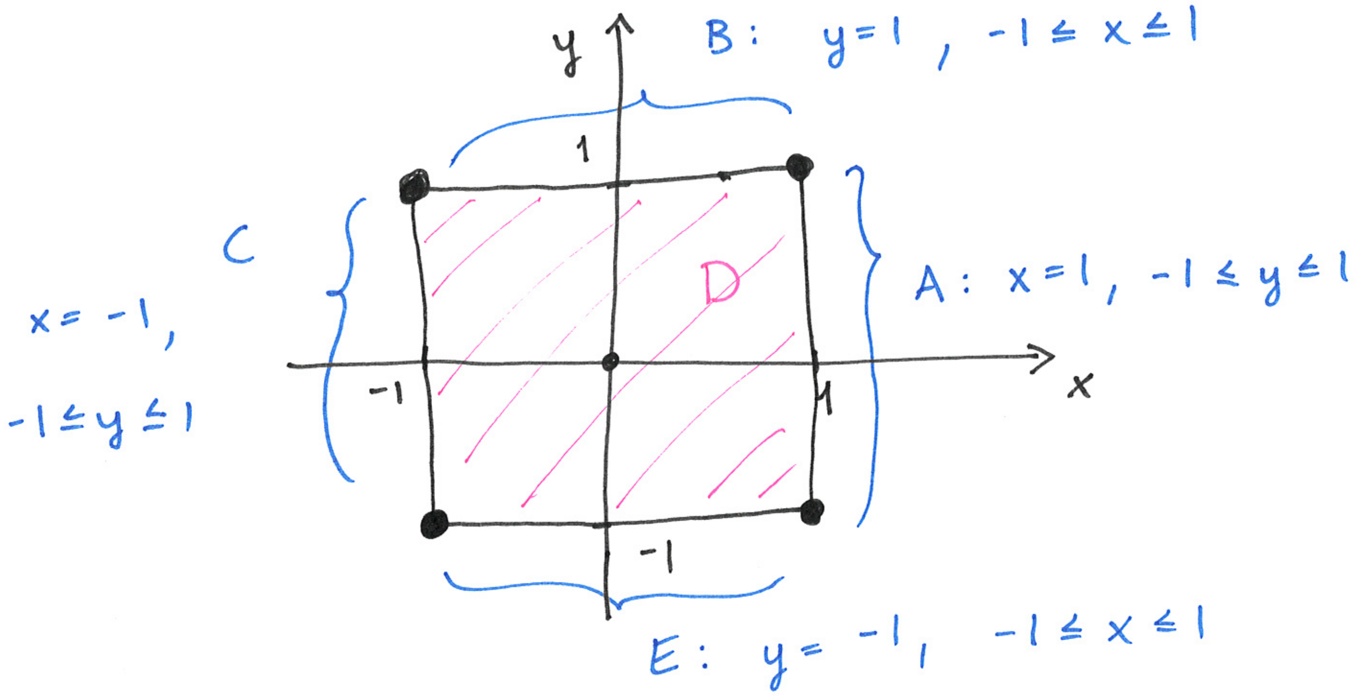
$$= x^3y - xy^3, \text{ when}$$



$$\underbrace{-1 \le x, y \le 1}_{D}$$

$-1 \le x \le 1$ and $-1 \le y \le 1$

Solution: Draw D:



NB: f is continuous, D is closed ($\leq$) and bounded (can be boxed in) $\Rightarrow$ D is compact

$\Rightarrow$ The extreme value theorem: f has a max. and min. (over D).

Candidate points

i) Interior stationary points:

$$f'_x = 0 \Rightarrow f'_x = 3x^2y - y^3 = 0$$

$$f'_y = 0 \Rightarrow f'_y = x^3 - 3xy^2 = 0$$

(1): $y(3x^2 - y^2) = 0$

(2): $x(x^2 - 3y^2) = 0$

From (1): $y = 0$ or $3x^2 - y^2 = 0 \rightarrow 3x^2 = y^2$

(2): $x(x^2 - 3\cdot 0^2) = 0$

$$x^3 = 0$$

$$\underline{x = 0}$$

(2): $x(x^2 - 3\cdot 3x^2) = 0$

$$x(x^2 - 9x^2) = 0$$

$$-8x^3 = 0$$

$$\underline{x = 0}$$

$$y^2 = 3x^2 = 3\cdot 0^2 = 0$$

$$\underline{y = 0}$$

Only one interior stationary point: $\underline{(x,y) = (0,0)}$

$$\Rightarrow \underline{f(0,0) = 0}$$

ii) Other interior critical points: f'_x and f'_y are def. everywhere $\Rightarrow$ No such pts.

iii) Boundary of D:

A: $x = 1, -1 \leq y \leq 1$:

$$f(x,y) = f(1,y) = y - y^3 =: h(y)$$

$:=$ define

$$h'(y) = 1 - 3y^2 = 0$$

$$3y^2 = 1$$

$$y^2 = \frac{1}{3}$$

$$y = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}} \quad \left(\text{Between } -1 \text{ and } 1\right)$$

Candidates from A: $\left(1, -\frac{1}{\sqrt{3}}\right), \left(1, \frac{1}{\sqrt{3}}\right), (1,1), (1,-1)$

Boundary pts. of A: One-variable

$f(1, -\frac{1}{\sqrt{3}}) = \dots = -\underline{\underline{\frac{2}{3\sqrt{3}}}}$, $f(1,1) = \underline{0}$

$f(1, \frac{1}{\sqrt{3}}) = \dots = \underline{\underline{\frac{2}{3\sqrt{3}}}}$, $f(1,-1) = \underline{\underline{0}}$

B : $y = 1, \ -1 \le x \le 1$:

$$f(x,1) = x(x^2-1) =: h(x)$$
$$= x^3 - x$$

$$h'(x) = 3x^2 - 1 = 0$$
$$x^2 = \frac{1}{3}$$
$$x = \pm \frac{1}{\sqrt{3}} \quad \text{(Between } -1 \text{ and } 1\text{)}$$

Candidates from B : $(-\frac{1}{\sqrt{3}}, 1)$, $(\frac{1}{\sqrt{3}}, 1)$,

$(1,1)$, $(-1,1)$

Boundary pts. of B

$f(-\frac{1}{\sqrt{3}}, 1) = \dots = \underline{\underline{\frac{2}{3\sqrt{3}}}}$, $f(1,1) = \underline{\underline{0}}$

$f(\frac{1}{\sqrt{3}}, 1) = \dots = -\underline{\underline{\frac{2}{3\sqrt{3}}}}$, $f(-1,1) = \underline{0}$

C: $x = -1, \ -1 \le y \le 1$

$$f(-1,y) = y^3 - y =: h(y)$$
$$h'(y) = 3y^2 - 1 = 0$$
$$y = \pm \frac{1}{\sqrt{3}} \quad \text{(Between } -1 \text{ and } 1\text{)}$$

Candidates from C: $\left(-1, -\frac{1}{\sqrt{3}}\right)$, $\left(-1, \frac{1}{\sqrt{3}}\right)$,

$(-1, 1)$, $(-1,-1)$

Boundary pts. of C

$f\left(-1, -\frac{1}{\sqrt{3}}\right) = \ldots = \frac{2}{3\sqrt{3}}$, $f(-1, 1) = 0$

$f\left(-1, \frac{1}{\sqrt{3}}\right) = \ldots = -\frac{2}{3\sqrt{3}}$, $f(-1,-1) = 0$

E : $y = -1$, $-1 \leq x \leq 1$

$$f(x, -1) = x - x^3 =: h(x)$$

$$h'(x) = 1 - 3x^2 = 0$$

$$x = \pm \frac{1}{\sqrt{3}} \quad \text{(Between } -1 \text{ and } 1\text{)}$$

Candidates from E: $\left(-\frac{1}{\sqrt{3}}, -1\right)$, $\left(\frac{1}{\sqrt{3}}, -1\right)$,

$(-1,-1)$, $(1,-1)$

Boundary pts. of E

$f\left(-\frac{1}{\sqrt{3}}, -1\right) = \ldots = -\frac{2}{3\sqrt{3}}$, $f(-1,-1) = 0$

$f\left(\frac{1}{\sqrt{3}}, -1\right) = \ldots = \frac{2}{3\sqrt{3}}$, $f(1,-1) = 0$

To conclude: Compare function values of the candidates ($4 \cdot 4 = 16$). Know, from EVT, that f has min.

and max. over D. Hence,

Maximum: $$f_{max} = \frac{2}{3\sqrt{3}}$$

at max. pts: $\left(1, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, 1\right)$, $\left(-1, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, 1\right)$

Minimum: $$f_{min} = -\frac{2}{3\sqrt{3}}$$

at min. pts: $\left(1, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, 1\right)$, $\left(-1, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, -1\right)$