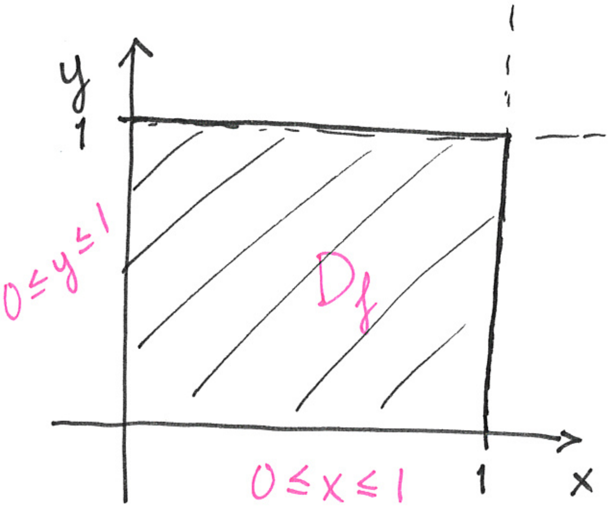
EBA 1180
Sect. 43
S25

Warm-up: $f(x,y) = x + y$,

$\underbrace{0 \leq x, y \leq 1}_{D_f}$

What are the (global) max./min. points?



See directly:

Max: $(1,1) \Rightarrow$

$f(1,1) = 1 + 1 = \underline{2}$

Min: $(0,0) \rightarrow$

$f(0,0) = 0 + 0 = \underline{0}$

NOTE: f has both (global) min. and max.

Q1: What if D_f: $0 < x, y < 1$?

Q2: What if min/max $f(x,y) = x + y$ over all of $\mathbb{R}^2$?

Constrained optimization and the extreme value theorem

- $f(x,y)$ is a continuous function on a set D in $\mathbb{R}^2$.

Extreme value theorem (EVT): If f is a continuous function on a compact set D in $\mathbb{R}^2$, then f has a maximum and a minimum on D.

Compact sets

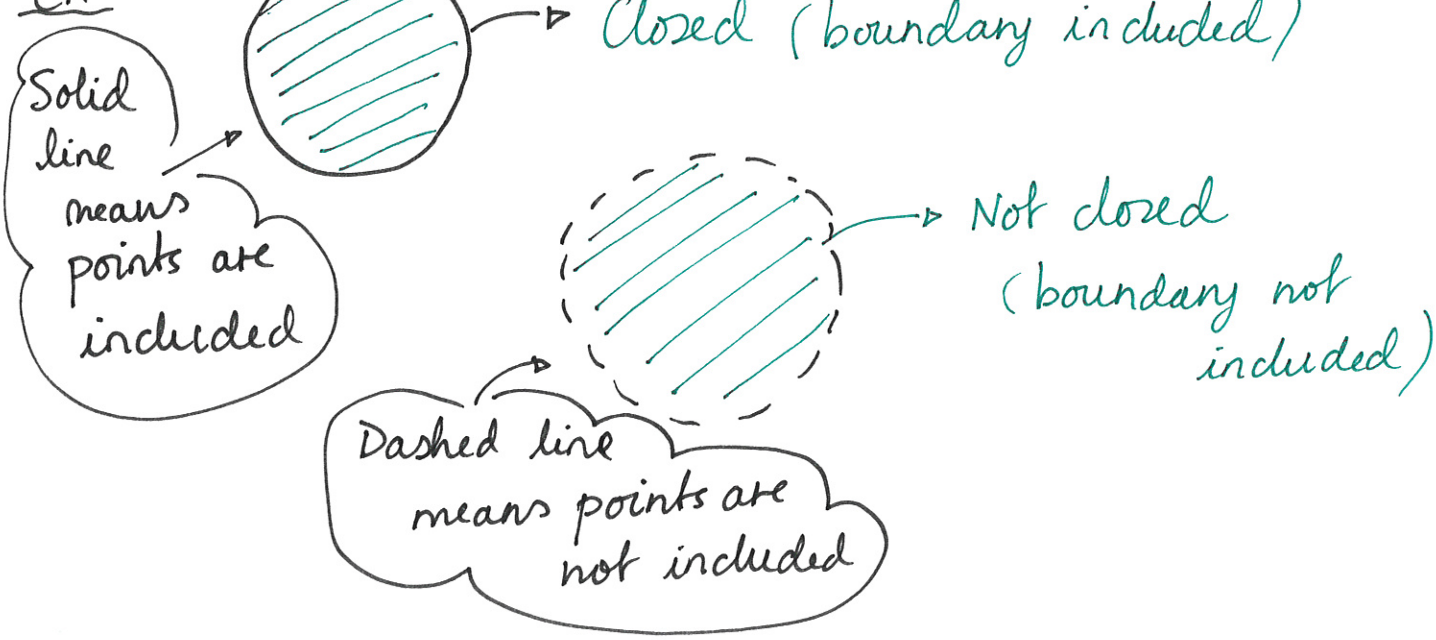
Def (Compact set): A subset D of $\mathbb{R}^2$ is compact if it is closed and bounded.

Def (Closed set): A subset D of $\mathbb{R}^2$ is closed if all boundary points of D are included in D

NOTE: $=$, $\leq$, $\geq$; closed

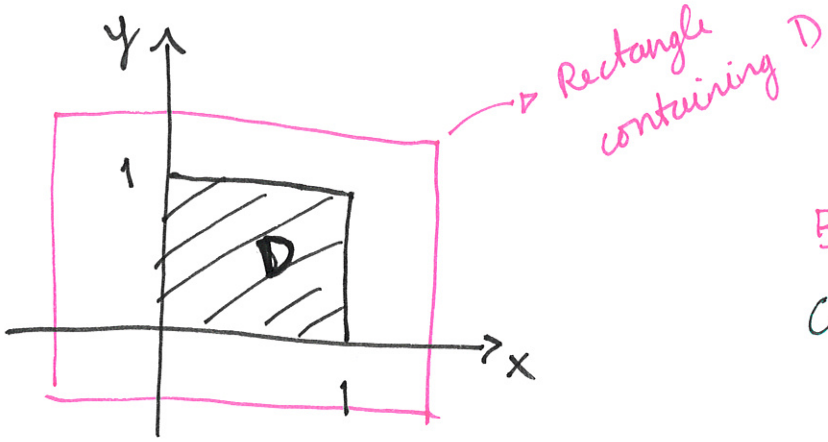
$<$, $>$; not closed

Ex:



Def (Bounded set): A subset D of $\mathbb{R}^2$ is bounded if there exists a rectangle in $\mathbb{R}^2$ (with finite side lengths) that includes all of D.

Ex:



$D: \quad 0 \leq x, y \leq 1$

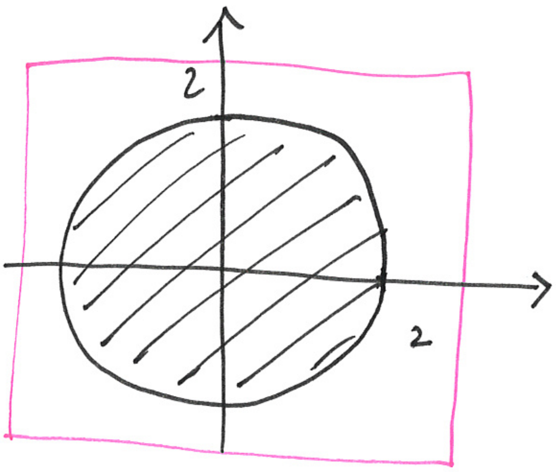
Bounded ✓

Closed ✓

$\Downarrow$

Compact ✓

Ex:



$D: \quad x^2 + y^2 \leq 4$ (circle, filled)

Q:

Bounded ✓

Closed ✓ (includes boundary)

$\Downarrow$

Compact ✓

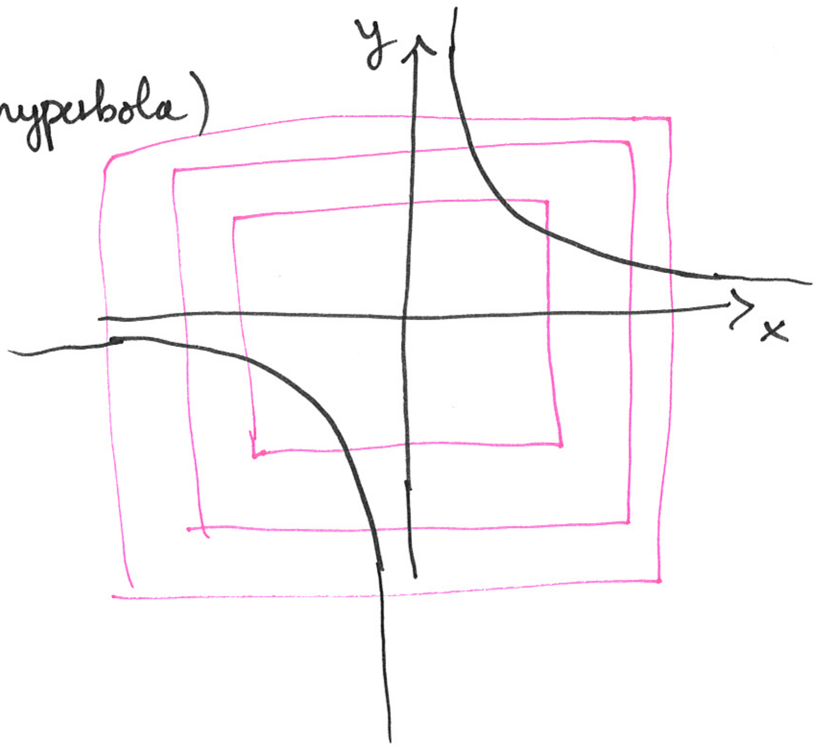
Ex: $D: \quad xy = 1$ (hyperbola)

$y = \frac{1}{x}$

Closed ✓

Bounded NO!

Compact NO!



Constrained optimization

$$\max/\min \ \underbrace{f(x,y) = x^2+y^2}_{\text{Objective function}} \quad \text{when} \quad \underbrace{0 \le x, y \le 1}_{\text{Constraints}}$$

→ Constrained optimization

$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x, y \le 1\}$, subset of $\mathbb{R}^2$

→ Set of admissible points

vs

$\max/\min \ f(x,y) = x^2+y^2$ } Unconstrained optimization

Unconstrained

max/min $f(x,y)$

Candidate pts

i) Stationary pts: $f'_x = 0$, $f'_y = 0$

ii) Other critical pts: f'_x or f'_y are not defined

iii) Boundary pts. of D_f:

(Local) classification:

Second derivative test (local max, local min, saddle pts)

Must check: Are any of these global max/min?

Constrained

max/min $f(x,y)$ when (x,y) in D

Candidate pts:

i) Interior stationary points:

$f'_x = 0$, $f'_y = 0$

ii) Other interior critical pts: f'_x or f'_y not defined.

iii) Boundary points of D; → ∂D; the boundary of D

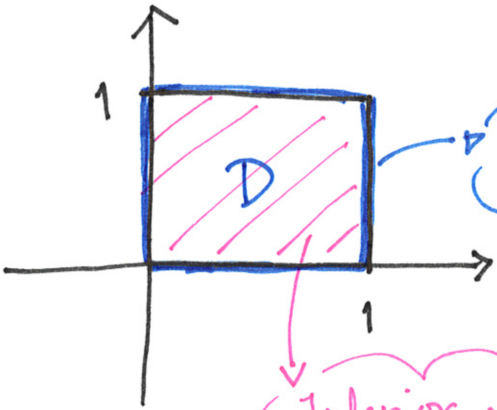
$D = \{(x,y) : \text{all constraints satisfied}\}$

EVT: If D is compact (closed and bounded), there is a global max and min.

→ Use EVT is D is compact to determine if candidates are global max/min

The boundary: ∂D

Ex: max/min $f(x,y) = x^2 + y^2$ when $0 \le x, y \le 1$



∂D = the boundary of D (the four sides of the square)

Interior of D

EVT: Has (global) max and min!

Max: Make x and y as large as possible $\Rightarrow$

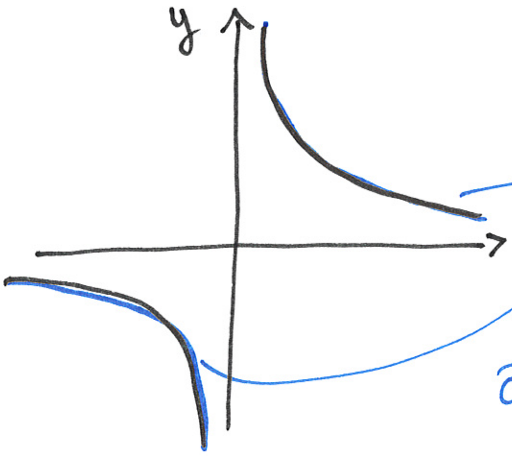
$x = y = 1 \Rightarrow f(1,1) = \underline{2}$

Min: Make x and y as small as possible $\Rightarrow$

$x = y = 0 \Rightarrow f(0,0) = \underline{0}$

Ex: D: $xy = 1$

$y = \frac{1}{x}$

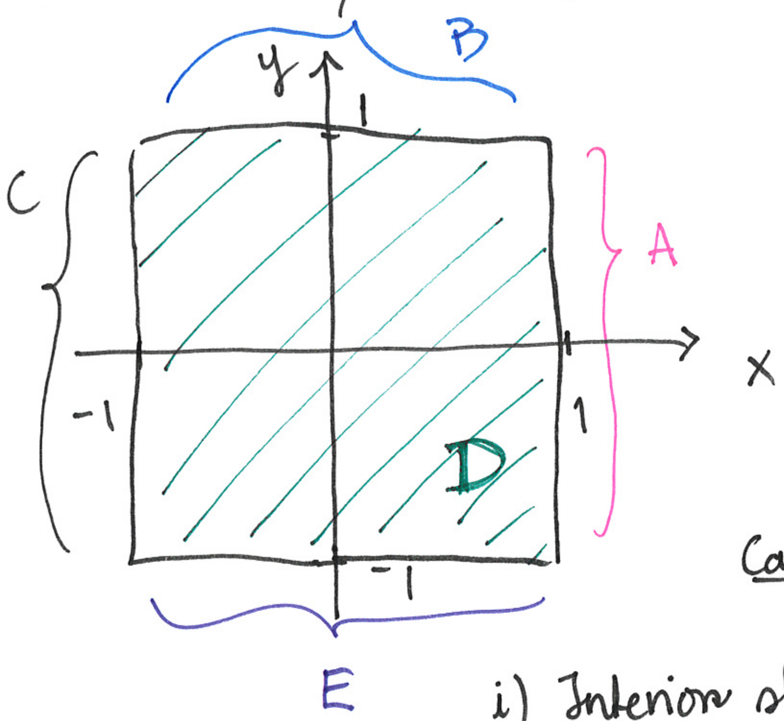


$D = \{(x,y) \in \mathbb{R}^2 : xy = 1\}$

∂D = boundary of D

= all points on D

Ex: max/min $f(x,y) = x^2 + y^2$ when $-1 \le x, y \le 1$



An example: Constrained optimization (technique)

Candidate pts:

i) Interior stationary pts:

$f'_x = 2x = 0 \Rightarrow x = 0$

$f'_y = 2y = 0 \Rightarrow y = 0$

$(0,0)$ is an interior point of D.

Candidate: $(x,y) = (0,0) \Rightarrow f(0,0) = 0^2 + 0^2 = 0$

ii) Other interior critical points: None.

iii) Boundary points of D: ∂D = four sides of square

$$\begin{cases} A: & x = 1, \quad -1 \le y \le 1 \\ B: & y = 1, \quad -1 \le x \le 1 \\ C: & x = -1, \quad -1 \le y \le 1 \\ E: & y = -1, \quad -1 \le x \le 1 \end{cases}$$

EVT : • f continuous ? OK!

• D compact ? Closed and bounded ?
Yes! ✓ ✓

EVT holds $\Rightarrow$ $\boxed{f(x,y) \text{ has a max and min}}$

$\Downarrow$

There is a max and min among the candidate points.

What is this max/min? Compare function values of candidates.

i) Stationary : $(0,0) \Rightarrow f(0,0) = 0$

ii) Critical : None.

iii) Boundary : A : $f(1,y) = 1 + y^2, \quad -1 \le y \le 1,$

max : $f(1,1) = f(1,-1) = 2$

min : $f(1,0) = 1$

Repeat for B, C and E (see additional note!)

Conclusion : D is compact, so there is a max/min from EVT. The highest function value among the candidates is : $f_{max} = 2$ → global max value

at the max points $(1,1), (1,-1), (-1,1), (-1,-1)$.

The lowest value among the candidates is:

$$f_{min} = 0$$ → global min value

at the min. pt. $(0, 0)$.