Course paper 2023

EBA 1180

Spring 25

Lect. 42

1) g) $\int_1^{e^2} \frac{\sqrt{\ln(x)}}{x}\,dx = \int \frac{\sqrt{u}}{\cancel{x}} \cancel{x}\,du$

SUBSTITUTION:

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$x = 1 \Rightarrow$

$u = \ln x$

$= \ln 1$

$= 0$

$x = e^2 \Rightarrow$

$u = \ln x$

$= \ln e^2 = 2$

$= \int_0^2 \sqrt{u}\,du$

$= \int_0^2 u^{\frac{1}{2}}\,du$

$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=0}^{2}$

$= \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$

$2^{\frac{3}{2}} = 2^{1+\frac{1}{2}}$

$= 2^1 \cdot 2^{\frac{1}{2}}$

$= 2\sqrt{2}$

$= \frac{2}{3}\left(2\sqrt{2} - 0\right)$

$= \underline{\underline{\frac{4}{3}\sqrt{2}}}$

Ex. text + general formula for parabola

2) P: $f(x) = a(x-2)^2 + 5$ → Vertex form

Parabola intersects the x-axis in $x = 2 \pm \sqrt{5}$: → Ex. text

$f(2 \pm \sqrt{5}) = 0$

From P: → $a(\pm\sqrt{5})^2 + 5 = 0$

$5a = -5$

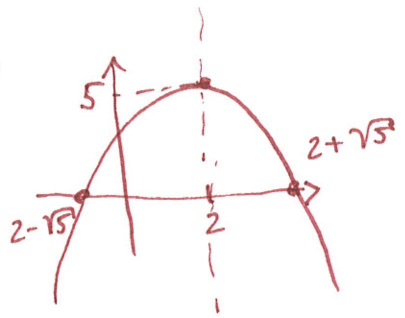
$\underline{a = -1}$

P: $f(x) = 5 - (x-2)^2 = 1 + 4x - x^2$

x-asympt. y-asympt.

General formula for hyperbola

H: $(x-0)(y-0) = c$



since $x=0$ and $y=0$ are asymptotes.

$$xy = c$$
$$y = \frac{c}{x}$$

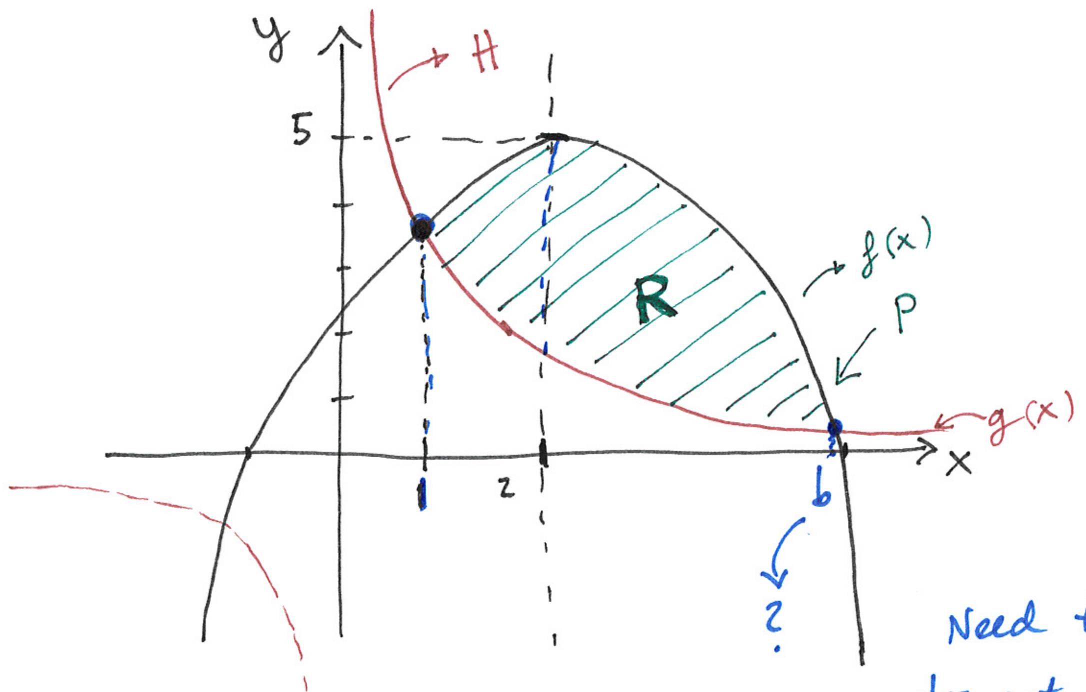
H: $g(x) = \frac{c}{x}$; What should c be?

parabola = hyperbola at $x=1$

Intersection in $x=1$: $f(1) \overset{!}{=} g(1)$

$$1 + 4 \cdot 1 - 1^2 = \frac{c}{1} \Rightarrow \underline{c = 4}$$

H: $\underline{\underline{g(x) = \frac{4}{x}}}$



Need to find b to get upper integration bound!

b) Area of R $= \int_1^b f(x) - g(x)\,dx$

Find b: Intersection: $f(x) = g(x)$ $(\sim)$

$$1 + 4x - x^2 = \frac{4}{x} \quad \Big| \cdot x$$

$$x + 4x^2 - x^3 = 4$$

$$x^3 - 4x^2 - x + 4 = 0$$

Know: f and g intersect at $x=1$ (ex. text), so $f(1) = g(1)$, so $(\sim)$ holds at $x=1$

Polynomial division:

$$(x^3 - 4x^2 - x + 4) : (x-1) = x^2 + \ldots$$
$$-(x^3 - x^2)$$
$$\ldots$$

$$x^3 - 4x^2 - x + 4 = (x-1)(x^2 - 3x - 4) = 0$$

$x = 1$ or $x^2 - 3x - 4 = 0$

$$(x-4)(x+1) = 0$$

$$x = 4, \quad x = -1$$

$\Rightarrow$ $b = 4$

From figure!

Area of R $= \displaystyle\int_1^4 \underbrace{1 + 4x - x^2}_{f(x)} - \overbrace{\frac{4}{x}}^{g(x)}\,dx$

$$= \left[x + 2x^2 - \frac{1}{3}x^3 - 4\ln|x| \right]_{x=1}^{4}$$

$$= \ldots \text{ plug in numbers } \ldots = \underline{\underline{12 - 8\ln 2}}$$

START 11.04

3) Total cash flow:

$$\int_0^{25} f(t)\,dt = \int_0^{25} 100\, e^{\sqrt{t}}\, dt$$

$$= \int_0^5 100\, e^u\, 2u\, du$$

SUBSTITUTION:

$u = \sqrt{t} = t^{\frac{1}{2}}$

$du = \frac{1}{2\sqrt{t}}\, dt$

$2\underbrace{\sqrt{t}}_{u}\, du = dt$

$2u\, du = dt$

BOUNDS:

$t = 0 \Rightarrow u = \sqrt{t} = \sqrt{0} = \underline{0}$

$t = 25 \Rightarrow u = \sqrt{t} = \sqrt{25} = \underline{5}$

$$= 200 \int_0^5 u\, e^u\, du$$

Int. by parts:

$\int u e^u du = ue^u - \int e^u \cdot 1\, du$

$\int w' v = wv - \int w v'$

$w' = e^u \Rightarrow w = e^u$

$v = u \Rightarrow v' = 1$

$$= 200 \left[u e^u - e^u \right]_{u=0}^{5}$$

$$= 200(5e^5 - e^5) - 200(0e^0 - e^0)$$

$$= \ldots = \underline{\underline{800\, e^5 + 200}}$$

Expression for net present value:

$$\int_0^{25} f(t)\, e^{-rt}\, dt = \int_0^{25} 100 e^{\sqrt{t}}\, \underline{\underline{e^{-rt}}}\, dt$$

6) b) $\cancel{x}\,\vec{v}_1 + y\,\vec{v}_2 + z\,\vec{v}_3 + \tilde{c}\,\vec{v}_4 = \vec{w}$

TRICK:

$$\begin{array}{c}\left[\begin{array}{cccc|c} 5 & 3 & 1 & 3 & a \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \end{array}\right] \\ \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array}\end{array} \quad -1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \end{array}\right] \quad -4 \quad -7$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & b-4(a-b) \\ 0 & -12 & 36 & 48 & c-7(a-b) \end{array}\right] \quad -\frac{12}{7}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & b-4(a-b) \\ 0 & 0 & 0 & 0 & \underbrace{c-7(a-b)-\frac{12}{7}\big(b-4(a-b)\big)}_{(\star)} \end{array}\right]$$

$\vec{w}$ is a lin. comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \iff$

the lin. syst. is consistent $\iff (\star) = 0$

$$\overbrace{c - 7(a-b) - \frac{12}{7}\left(b - 4(a-b)\right)}^{(\star)} = 0 \quad \Big| \cdot 7$$

$$7c - 49(a-b) - 12b + 48(a-b) = 0$$

$$7c - 12b - (a-b) = 0$$

$$-a - 11b + 7c = 0 \quad \Big| \cdot (-1)$$

minus!

$$a + 11b - 7c = 0$$

Conclusion: $\vec{w}$ is a lin. comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$

$\iff a + 11b - 7c = 0.$

7.) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $X = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$

X must be 2×2 for AX and XA to be def.

$AX = XA$

$$AX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

For $AX = XA$:

$$\left.\begin{aligned} c &= b \\ d &= a \\ a &= d \\ b &= c \end{aligned}\right\} \quad c, d \text{ free, } a = d \text{ and } b = c$$

$$(a,b,c,d) = (d,c,c,d) = c\,(0,1,1,0) + d\,(1,0,0,1)$$

where c and d are free.

Conclusion: $X = c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} d & c \\ c & d \end{bmatrix}$$

where c and d are free.