Interpretation of partial derivatives

What does $f_x'(a,b)$ and $f_y'(a,b)$ mean?

Ex: $f(x,y) = x^3 - 3xy + y^3$

$\Rightarrow f_x'(x,y) = 3x^2 - 3y \, , \quad f_y'(x,y) = -3x + 3y^2$

Let $(x,y) = (2,1)$. Then:

$$f(2,1) = 2^3 - 3\cdot 2\cdot 1 + 1^3 = \underline{3}$$

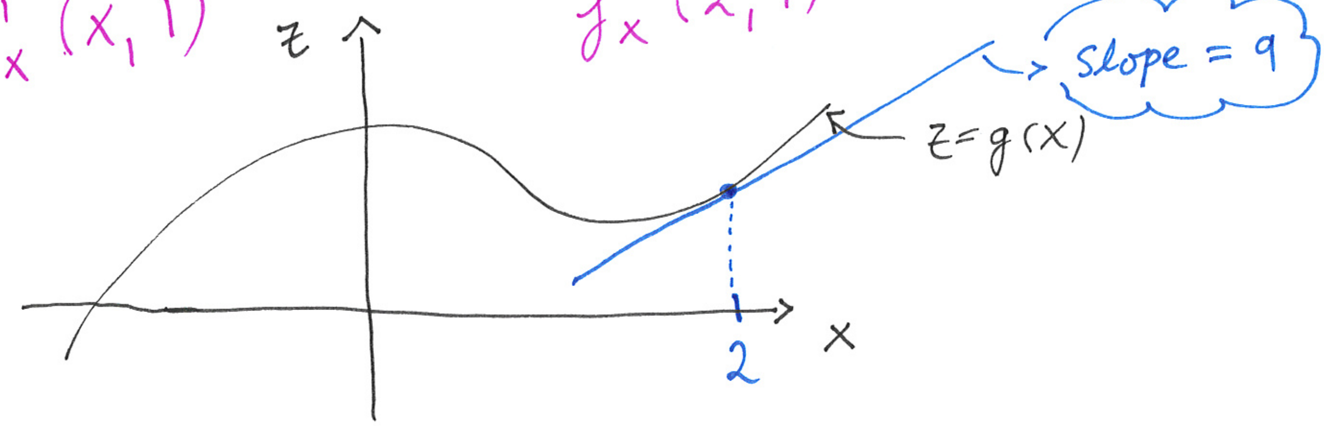
$$f_x'(2,1) = 3\cdot 2^2 - 3\cdot 1 = \underline{9}$$

$$f_y'(2,1) = -3\cdot 2 + 3\cdot 1^2 = \underline{-3}$$

In the x-direction $(y=1)$:

$$f(x,1) = x^3 - 3x\cdot 1 + 1^3 = x^3 - 3x + 1 \overset{\text{Defined as}}{=:} g(x)$$

$$\underbrace{g'(x)}_{f_x'(x,1)} = 3x^2 - 3 \, , \quad \underbrace{g'(2)}_{f_x'(2,1)} = 3\cdot 2^2 - 3 = \underline{9} \quad \text{SAME!}$$

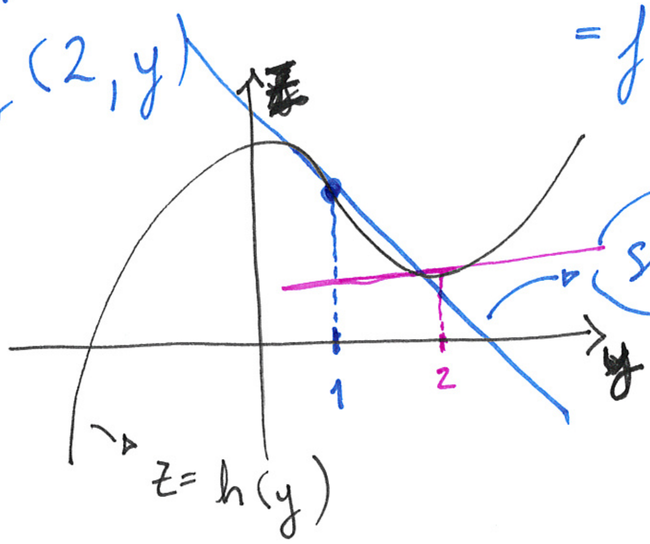


In the y-direction $(x = 2)$:

$$f(2, y) = 2^3 - 3\cdot 2y + y^3 = 8 - 6y + y^3 =: h(y)$$

$$\Rightarrow \underbrace{h'(y)}_{= f'_y(2,y)} = -6 + 3y^2, \text{ so } \underbrace{h'(1)}_{= f'_y(2,1)} = -6 + 3\cdot 1^2 = \underline{-3}$$

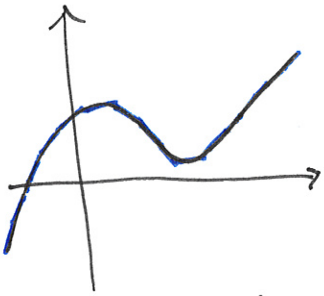
Same as pg. 1



Slope : -3

Linear approximation of $f(x,y)$ at (x_0, y_0):

Tangent plane of f at (x_0, y_0)



$$L(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)(x - x_0) + f'_y(x_0,y_0)(y - y_0)$$

Plane: $\underset{f'_x(x_0,y_0)}{a}x + \underset{f'_y(x_0,y_0)}{b}y + c$

The gradient

Def (gradient): The gradient of $f(x,y)$ is

$$\underbrace{\nabla f}_{\text{"The gradient of } f\text{"}} = \underbrace{\begin{bmatrix} f'_x \\ f'_y \end{bmatrix}}_{\text{a vector}}$$

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

$$\nabla f = \begin{bmatrix} 2x - 2 \\ 2y + 4 \end{bmatrix}$$

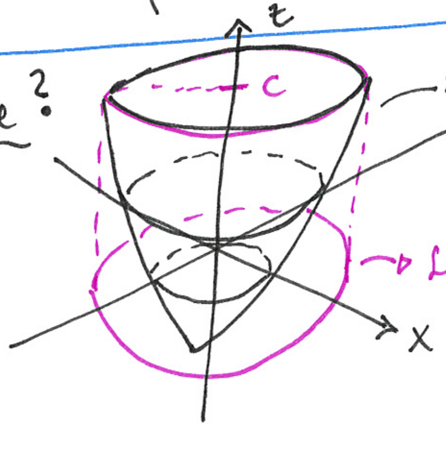
The gradient of f in $(x,y) = (-2,2)$:

$$\nabla f(-2,2) = \begin{bmatrix} 2\cdot(-2) - 2 \\ 2\cdot 2 + 4 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -6 \\ 8 \end{bmatrix}}}$$

RESULT: The gradient at a point is a normal vector to the tangent line of the level curve at that point.

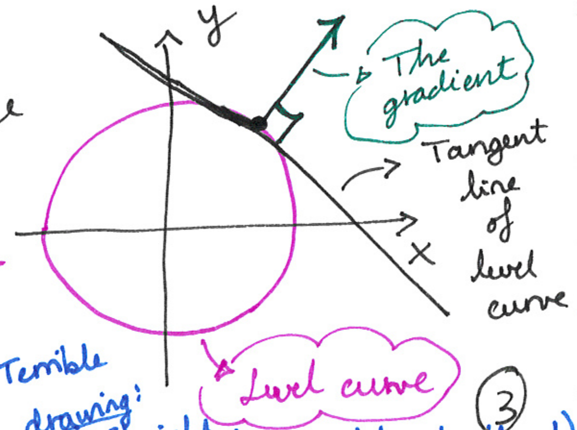
Where are we?

From Tuesday: $f(x,y)$ from EX.



Surface: xyz-plane

Level curve: xy-plane



(Terrible drawing: Straight line + Not cut through)

START: 13.01

Why does the result hold?

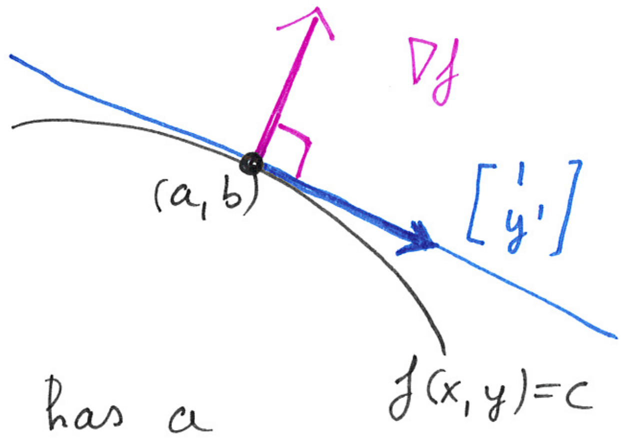
$$f(x,y) = c$$

Implicit differentiation: $\quad f'_x + f'_y\, y' = 0$

w.r.t. x: $y = y(x)$

Write as an inner product:

$$\underbrace{\begin{bmatrix} 1 \\ y' \end{bmatrix}}_{\text{Vector in the direction of tangent of level curve}} \cdot \underbrace{\begin{bmatrix} f'_x \\ f'_y \end{bmatrix}}_{\nabla f} = 0$$



So: $\nabla f(a,b)$ is a vector which has a 90° angle on the tangent of the level curve in the point (a,b).

Ex ctd.: $\quad \nabla f(-2,2) = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$

From Tuesday's lecture

Tangent line of the level curve at $(-2,2)$:

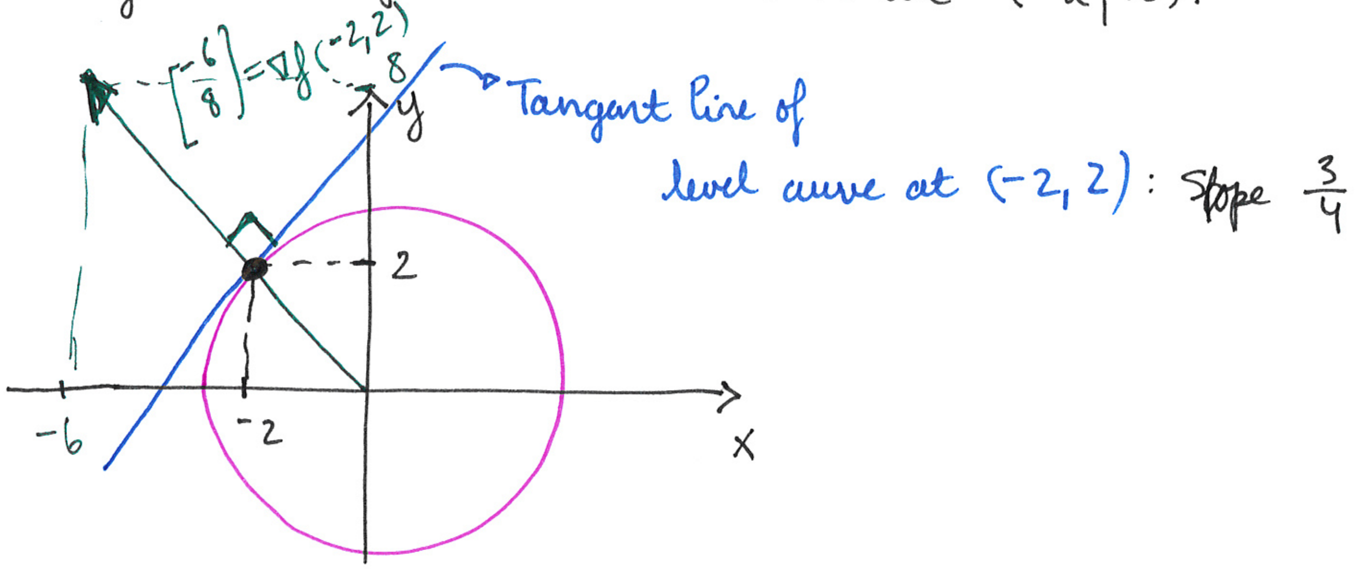
$$y = \frac{3}{4}x + \frac{7}{2}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{3}{4}x + \frac{7}{2} \end{bmatrix} = \begin{bmatrix} x \\ \frac{3}{4}x \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{7}{2} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}_{\text{direction of the tangent line of the level curve}} x + \begin{bmatrix} 0 \\ \frac{7}{2} \end{bmatrix}$$

$$\nabla f(-2,2) \cdot \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = -6 + \cancel{8}^{2} \cdot \frac{3}{\cancel{4}}$$

$$= -6 + 6 = \underline{0}$$

So: $\nabla f(-2,2) \perp \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}$ ($90°$ angle), i.e., the tangent line of the level curve at $(-2,2)$.



Directional derivative

Def (Directional derivative): Let $f(x,y)$ be a function, $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ a 2-vector. Then,

$$f'_{\vec{a}} := \vec{a} \cdot \nabla f$$

"The directional derivative of f wrt. $\vec{a}$"

dot product; a number

Ex: $f(x,y)$ as before, $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$f'_{\vec{a}} = \vec{a} \cdot \nabla f = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2x-2 \\ 2y+4 \end{bmatrix}$$

$$= 2(2x-2) + 1 \cdot (2y+4)$$

$$= \dots = \underline{4x + 2y} \to \text{A number}$$

Can insert a point: Say: $(1,1)$

$$f'_{\vec{a}}(1,1) = 4 \cdot 1 + 2 \cdot 1 = \underline{\underline{6}}$$

More about the second derivative test: Optimization

NOTE: If (x^*, y^*) is a stationary point with

$$H(f)(x^*,y^*) = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

$A \to f''_{xx}$, $B \to f''_{xy}$, $B \to f''_{yx}$, $C \to f''_{yy}$

Then: ① If $\underbrace{AC - B^2}_{\text{det. of Hessian}} > 0$, then $AC > B^2 \geq 0$

det. of Hessian: Local max. or local min.

Since square

$\Rightarrow AC > 0$. But then, $\to$ trace

$+ \cdot + = +$ i) $A, C > 0$: $A + C > 0$ Local min. from 2nd deriv. test

$- \cdot - = +$ ii) $A, C < 0$: $A + C < 0$ Local max from 2nd deriv. test

Hence, the possible cases are:

i) $A, C > 0$; local min.

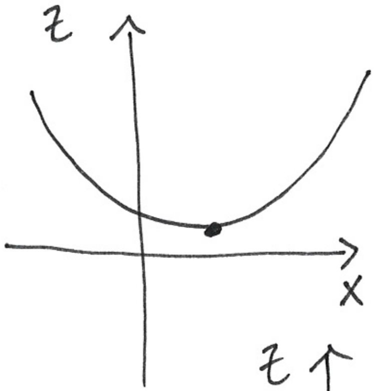
ii) $\underbrace{A}_{f''_{xx}}, \underbrace{C}_{f''_{yy}} < 0$; local max.

<u>Recall</u>: Second order derivative ≥ 0 ; $\boxed{\text{Convex}}$

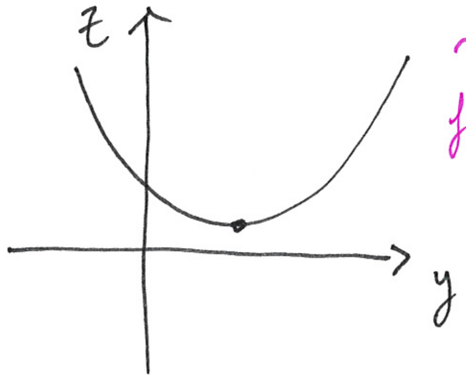
" ≤ 0 ; Concave

<u>Graphically</u>: Cuts of the graph $z = f(x,y)$

i) $A, C > 0$: Locally convex cuts



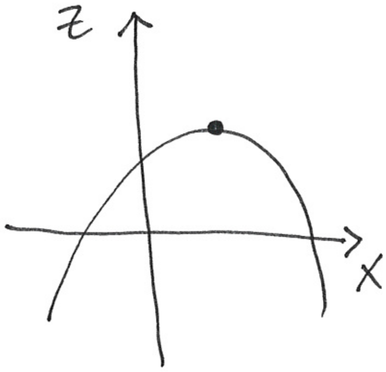
$\underbrace{f''_{xx}}_{A} > 0$



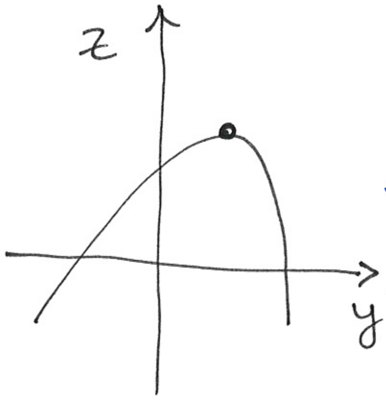
$\overbrace{f''_{yy}}^{C} > 0$

Local minimum
(locally convex)

ii) $A, C < 0$: Locally concave cuts



$\underbrace{f''_{xx}}_{A} < 0$



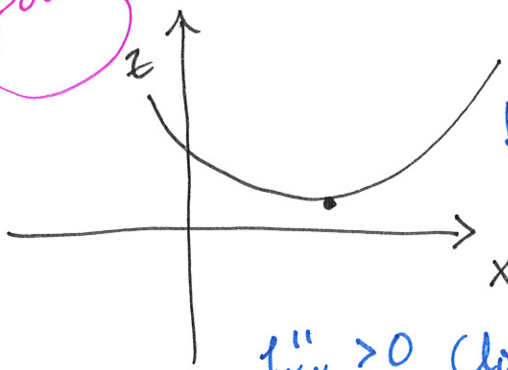
$\underbrace{f''_{yy}}_{C} < 0$

Local max.
(locally concave)

(2) If $\underbrace{AC - B^2}_{\det H(f)} < 0$; a typical case is

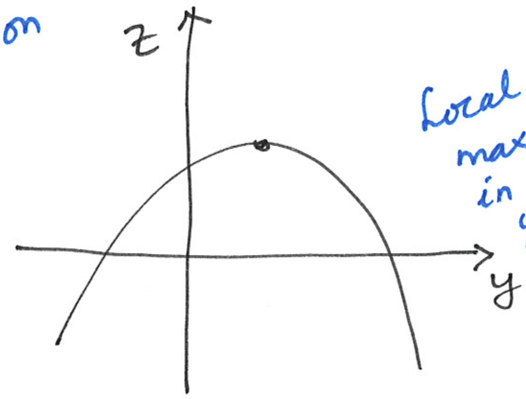
$$\underbrace{A}_{f''_{xx}} > 0 \;, \quad \underbrace{C}_{f''_{yy}} < 0$$

Saddle point



Local min. in x-direction

$f''_{xx} > 0$ (loc. convex)



Local max in y-dir.

$f''_{yy} < 0$ (loc. concave)