STATIONARY POINT CTD.

EBA1180
Spring 25
Lect. 40

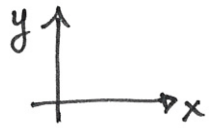
Ex:

$$f(x,y) = \underbrace{x^3 + 3xy + y^3}_{}\,,$$

Partial derivatives defined everywhere: No points where either f'_x or f'_y are not defined.

$\underbrace{D_f}_{} = \mathbb{R}^2$ $(\mathbb{R} \times \mathbb{R})$

No boundary points to consider



stationary points (also the candidate points for max/min)

FOC (first order conditions)

$$\begin{cases} f'_x = 3x^2 + 3y = 0 \\ f'_y = 3x + 3y^2 = 0 \end{cases}$$

$$\begin{cases} x^2 + y = 0 & \Rightarrow \quad y = -x^2 \quad (*) \\ x + y^2 = 0 & \longrightarrow \quad x + (-x^2)^2 = 0 \end{cases}$$

$$x + x^4 = 0$$

$$x(1 + x^3) = 0$$

$\underline{x = 0}$:

From (*): $y = -0^2 = \underline{0}$

$\underline{1 + x^3 = 0}$:

$$x^3 = -1$$

$$x = \sqrt[3]{-1} = \underline{-1}$$

From (*): $y = -(-1)^2 = \underline{-1}$

So the stationary points are $(0,0)$ and $(-1,-1)$. Since there are no boundary points or points where either partial derivative is not defined, these are also the candidate points.

The second derivative test

Ex ctd. : $f(x,y) = x^3 + 3xy + y^3$

$f'_x = 3x^2 + 3y$

$f'_y = 3x + 3y^2$

Candidate pts:

$(x^*, y^*) = (0,0), (-1,-1)$

$f(0,0) = 0^3 + 3 \cdot 0 \cdot 0 + 0^3 = \underline{0}$

$f(-1,-1) = (-1)^3 + 3 \cdot (-1) \cdot (-1) + (-1)^3 = -1 + 3 - 1 = \underline{1}$

Classify the candidate points using the second derivative test

Hessian:

$$H(f) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} 6x & 3 \\ 3 & 6y \end{bmatrix}$$

Candidate point $(0,0)$:

$$H(f)(0,0) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \Rightarrow$$

$\det H(f)(0,0) = 0 \cdot 0 - 3 \cdot 3 = -9 < 0$

$\Rightarrow$ $(0,0)$ is a saddle point for f.

Candidate point $(-1,-1)$:

$$H(f)(-1,-1) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$$

$$\det H(f)(-1,-1) = 36 - 9 = 27 > 0$$

since >0: Need to compute trace to classify

$$\operatorname{tr} H(f)(-1,-1) = -6 + (-6) = -12 < 0$$

$$\Downarrow$$

$(-1,-1)$ is a local max. for f

Global max/min.?

Conclusion: $f(x,y) = x^3 + 3xy + y^3$ has no minimum. — " — has a local maximum $f(-1,-1) = 1$ at $(-1,-1)$.

NOTE: $f(10,10) = 10^3 + 3\cdot 10 \cdot 10 + 10^3$

$= 2300 > 1$

$= f(-1,-1)$; value in local max

$\Rightarrow$ $(-1,-1)$ is not a global max. for f.

$\Rightarrow$ f has no global max.

Tangents of level curves

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

Level curve: All (x,y) s.t. $f(x,y) = c$.

$$x^2 - 2x + y^2 + 4y = c$$

TRICK: Complete the squares:

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) = c + 1 + 4$$

$$(x-1)^2 + (y+2)^2 = c + 5 \quad : (\star)$$

START: 11.03

$c+5>0$:

So $c > -5$

$\Rightarrow$ Level curve is a circle with center in $(1,-2)$ and $r = \sqrt{c+5}$

$c+5=0$

So $c = -5 \Rightarrow$ Level "curve" is $(1,-2)$ WHY?

Need $\begin{cases} x-1=0 \\ y+2=0 \end{cases}$

because sum of squares is $0 \Rightarrow$ Squares must be 0

$\Rightarrow x-1=0$
$y+2=0$

$c+5<0$:

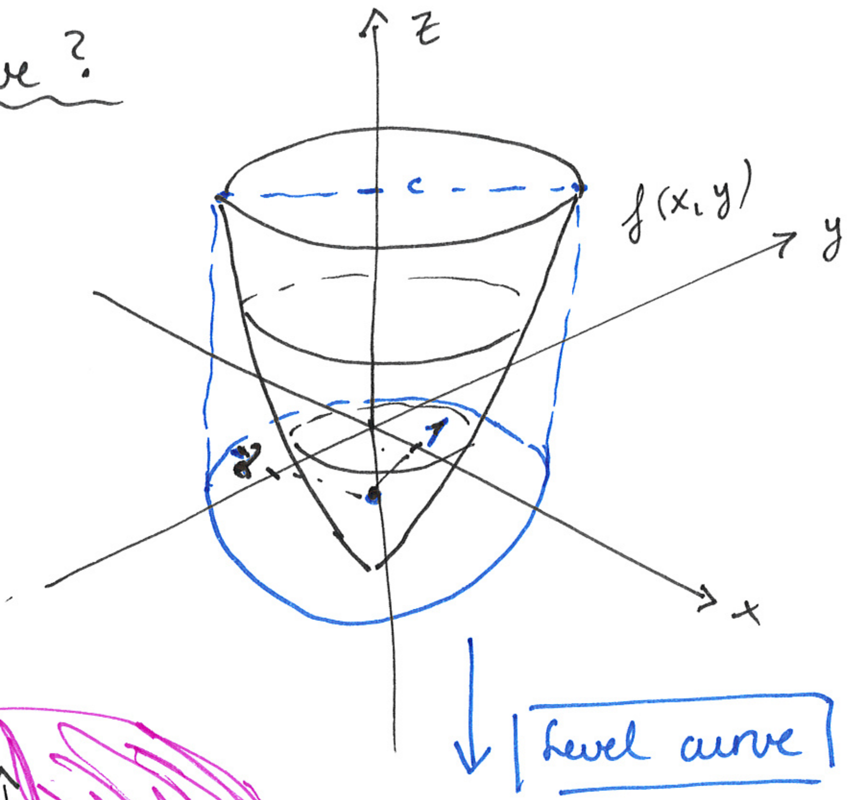
So $c < -5 \Rightarrow$

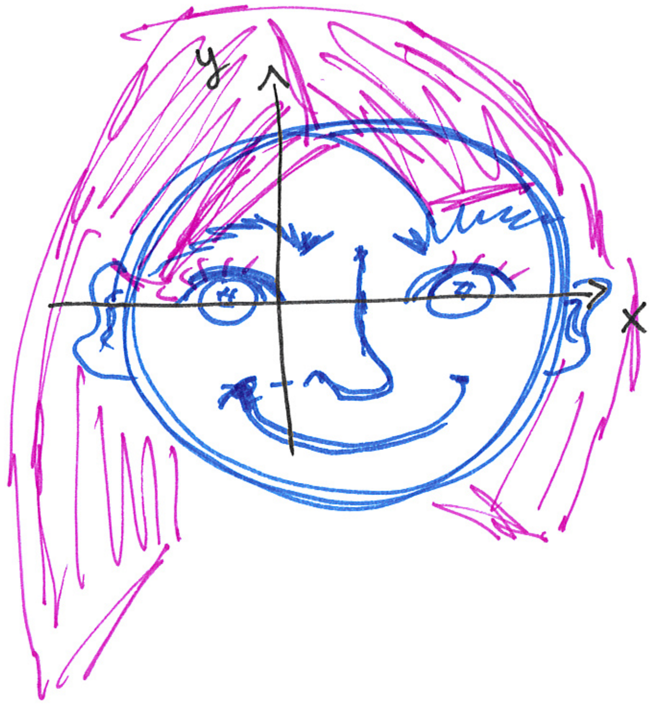
$(\star)$ never holds (sum of squares never negative)

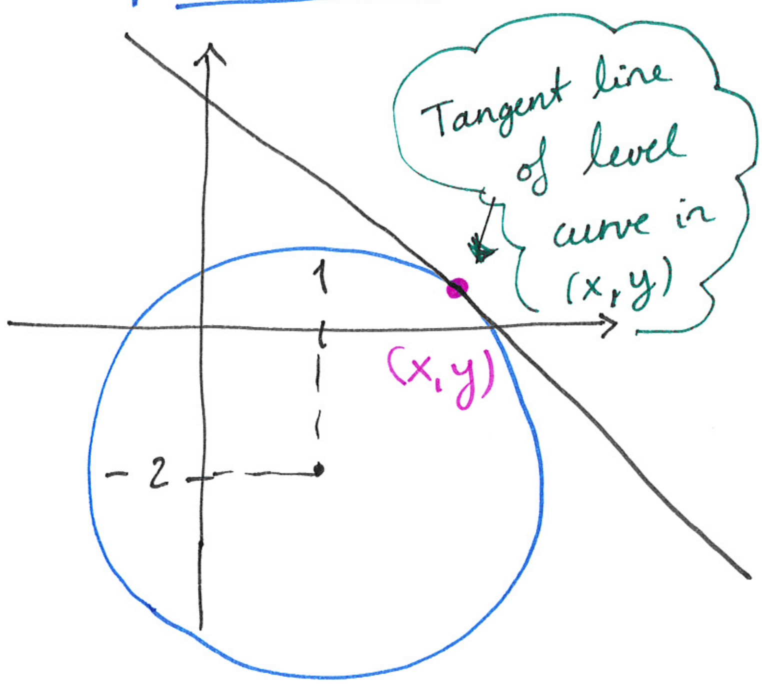
$\Rightarrow$ No level curve, i.e. f never attains a value $c < -5$.

How to find the tangent line of a level curve in a point (x,y)?

Where are we?







Ex. ctd: Tangent line of $\underbrace{\text{level curve}}_{\text{of } f}$ in $(x,y) = (-2, 2)$?

$$f(x,y) = x^2 - 2x + y^2 + 4y$$

1) Find level of level curve corresponding to the point:

$$z = f(-2,2) = (-2)^2 - 2\cdot(-2) + 2^2 + 4\cdot 2$$

$$= 4 + 4 + 4 + 8 = 20$$

$$\Rightarrow c = 20$$

Level curve at level 20 corresponds to $(x,y) = (-2,2)$.

2) Point - slope formula:

slope: unknown

$$y - 2 = k\,(x - (-2))$$

$$y = k(x+2) + 2 \quad : \quad (\sim)$$

What is the slope k?

3) Implicit differentiation: Think $y = y(x)$:

$$\big(\underbrace{x^2 - 2x + y^2 + 4y}_{f(x,y)}\big)'_x = (\underbrace{20}_{c})'_x$$

$$2x - 2 + \underbrace{2y(x)\,y'(x)}_{\text{CHAIN RULE}} + 4y'(x) = 0$$

$$2x - 2 + 2yy' + 4y' = 0$$

Solve for y': $y' = -\dfrac{2x-2}{2y+4} = -\dfrac{f'_x}{f'_y}$

OBSERVE

y' = the slope since y is lin. func. of x

$y = kx + b$

$y' = k$; the slope

4) Insert $(x,y) = (-2,2)$:

$$y'\Big|_{(-2,2)} = -\frac{2\cdot(-2)-2}{2\cdot 2+4} = \underbrace{\frac{3}{4}}$$

the slope of the tangent line in $(-2,2)$

Hence, from $(\sim)$:

$$y = \frac{3}{4}(x+2) + 2$$

$$= \frac{3}{4}x + \frac{3}{2} + 2 = \underline{\underline{\frac{3}{4}x + \frac{7}{2}}}$$

Result: If $f(x,y) = c$, then

$$f'_x + f'_y\, y' = 0. \text{ Hence,}$$

$$\underbrace{y'}_{\text{slope}} = -\frac{f'_x}{f'_y}$$

Implicit differentiation + chain rule (as above)

HOLDS IN GENERAL