Graphs and level curves

EBA 1180
Spring 25
Lect. 39

<u>Ex</u>: $f(x,y) = x^2 + y^2$

<u>Level curves</u>: $\rightarrow$ $f(x,y) = c$

<u>$c=2$</u>: $f(x,y) = 2$
$x^2 + y^2 = 2 \rightarrow$ Circle, center in $(0,0)$, $r = \sqrt{2}$

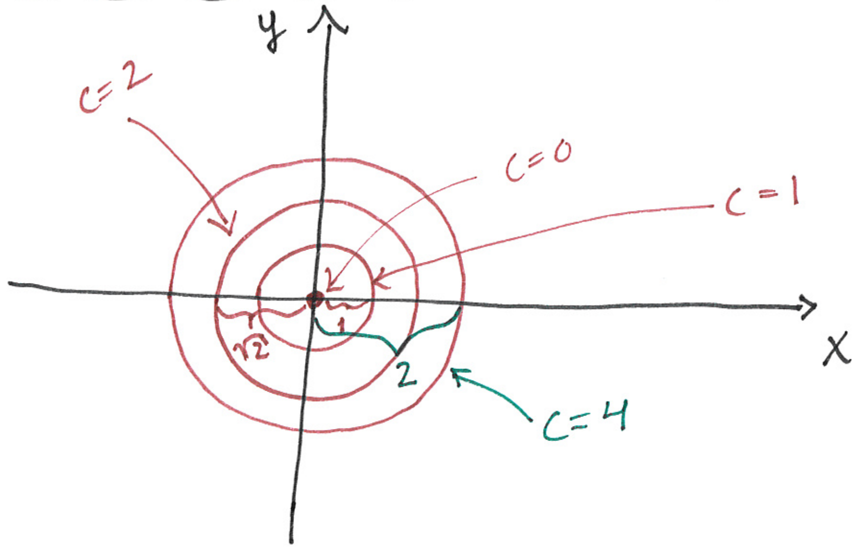
<u>$c=1$</u>: $f(x,y) = 1$
$x^2 + y^2 = 1 \rightarrow$ Circle, center in $(0,0)$, $r = 1$

<u>$c=0$</u>: $f(x,y) = 0$
$x^2 + y^2 = 0 \iff x = y = 0$; Level "curve" is just a point $(0,0)$

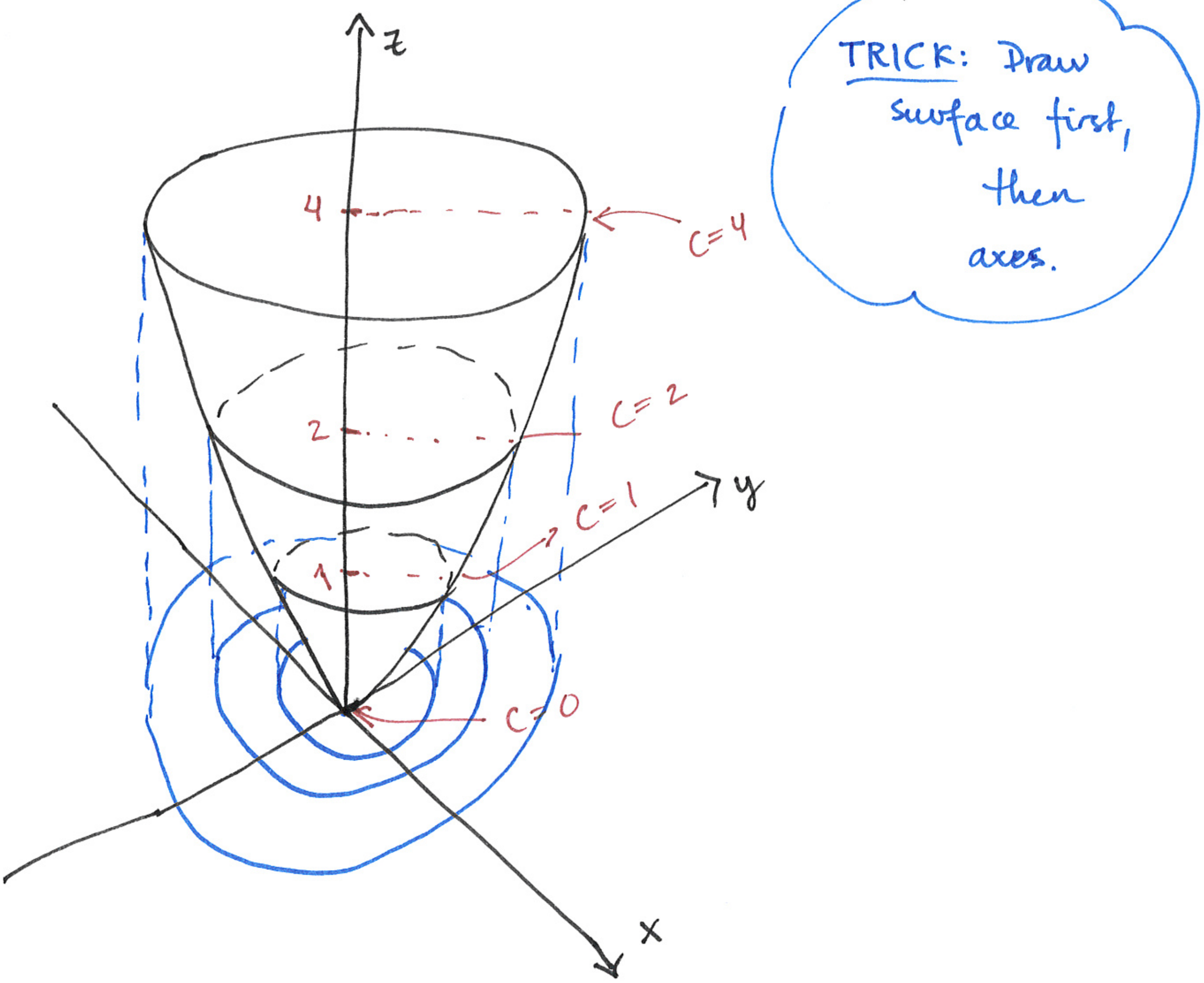
<u>$c=4$</u>: $f(x,y) = 4$
$x^2 + y^2 = 4 \rightarrow$ Circle, center in $(0,0)$, $r = \sqrt{4} = 2$

<u>$c=-1$</u>: $f(x,y) = -1$
$x^2 + y^2 = -1 \rightarrow$ No such points

<u>Illustration of level curves from above</u>: In xy-plane

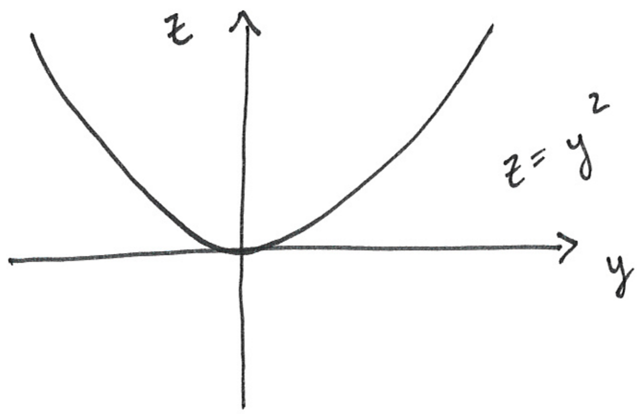


Use level curves to draw graph of f:

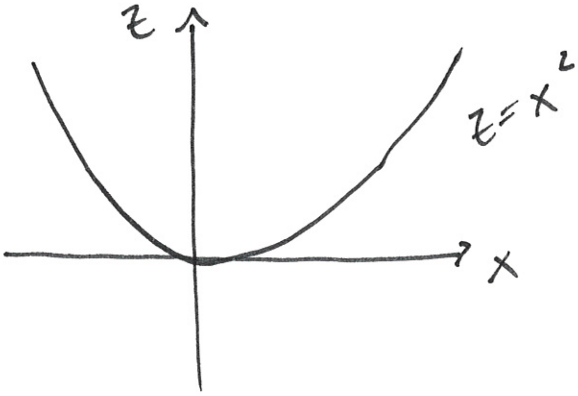


Q: If $x=0$, what does $z = f(x,y) = f(0,y)$ look like?

Cut $x=0$: $z = f(0,y) = 0^2 + y^2 = y^2$



Cut $y=0$: $z = f(x,0) = x^2 + 0^2 = x^2$



Linear functions

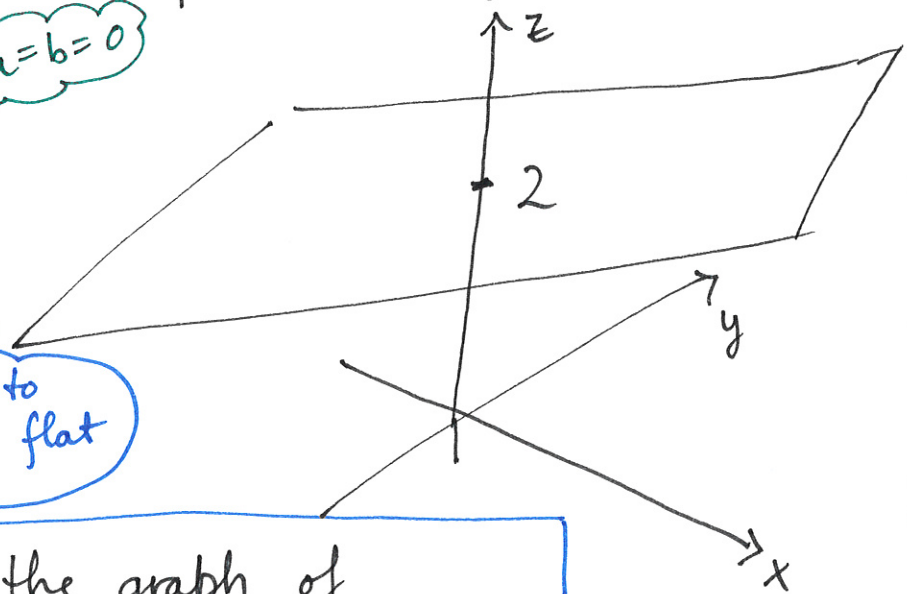
Def (Linear function): A function in two variables is linear if it can be written

$$f(x,y) = ax + by + c$$

FACT: The graph of f is a plane $\Leftrightarrow$ f is linear.

Ex: $f(x,y) = 2 \rightarrow a=b=0$

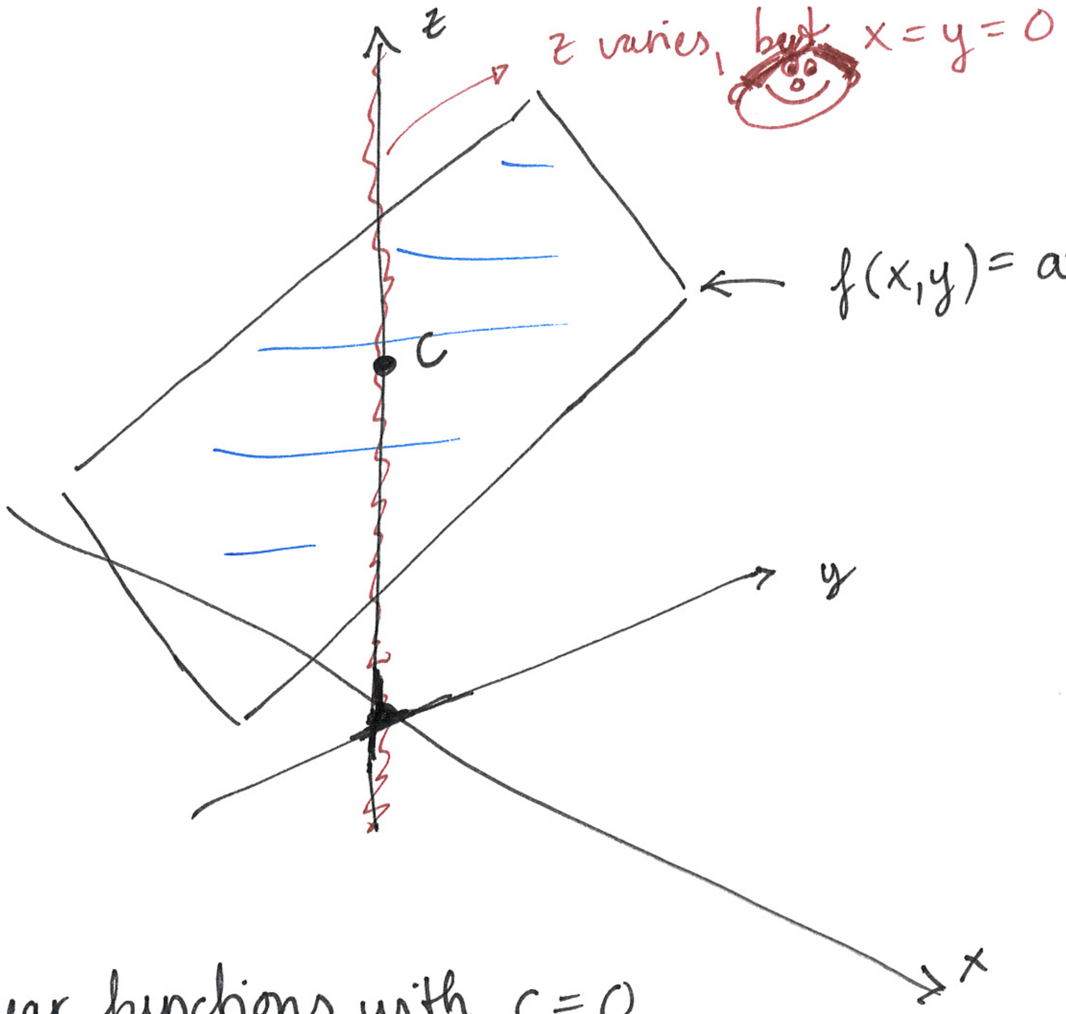
$z = f(x,y) = 2$



Supposed to lay flat

NB: The intersection of the graph of $f(x,y) = ax + by + c$ and the z-axis is $z=c$.

$x=y=0 \Rightarrow f(0,0) = a \cdot 0 + b \cdot 0 + c = c$



$f(x,y) = ax + by + c$

Linear functions with $c = 0$

$$f(x,y) = ax + by$$

$$z = ax + by$$

$$0 = ax + by - z \iff \begin{bmatrix} a \\ b \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} a \\ b \\ -1 \end{bmatrix} \perp \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence, the graph of $f(x,y) = ax + by$: All vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that are normal to $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$.

90° angle

This is a plane and $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ is its normal vector.

Ex: $f(x,y) = x - 2y$

$z = x - 2y$

$0 = x - 2y - z$

$\Rightarrow$ The plane that is the graph of $f(x,y)$ has normal vector $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

Conclusion: The graph of a linear function in two variables $f(x,y) = ax + by + c$ is a plane with normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ and intersection with the z-axis $z = c$.

Partial derivatives of functions in two variables

Ex: $f(x,y) = 3x + 4y - 5$

$f(x,y) = x^2 + y^2$

Partial derivatives:

$$\underbrace{f'_x(x,y)}_{\text{"Partial derivative of } f \text{ w.r.t. } x\text{"}} \overset{\text{is defined}}{:=} \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

TO COMPUTE: Think of y as a constant. Use normal rules for differentiation to find f'_x.

$$f'_y(x,y) := \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

COMPUTE: Think of x as a constant.

Ex: i) $f(x,y) = 3x + 4y - 5$

$$f'_x(x,y) = 3 + 0 - 0 = \underline{\underline{3}}$$

$$f'_y(x,y) = 0 + 4 - 0 = \underline{\underline{4}}$$

ii) $f(x,y) = x^2 + y^2$

$$f'_x(x,y) = 2x + 0 = \underline{\underline{2x}}$$

$$f'_y(x,y) = 0 + 2y = \underline{\underline{2y}}$$

Double derivatives

$f''_{xx}(x,y) = 2$, $f''_{yy}(x,y) = 2$

$f''_{xy}(x,y) = 0$, $f''_{yx}(x,y) = 0$

NB: Cross derivatives

First x, then y

First y, then x

SAME

Def (Stationary point): Let $f(x,y)$ be a function. A point $(x,y) = (a,b)$ is a stationary point for f if:

$$f'_x(a,b) = 0 = f'_y(a,b)$$

To find stationary points: Solve the system of equations

Solve for (x,y)
$$\begin{cases} f'_x(x,y) = 0 \\ f'_y(x,y) = 0 \end{cases}$$

The Hessian of $f(x,y)$

Def (Hessian): The Hessian of $f(x,y)$ is the 2×2 matrix

$$\mathcal{H}(f)(x,y) := \begin{bmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{bmatrix}$$

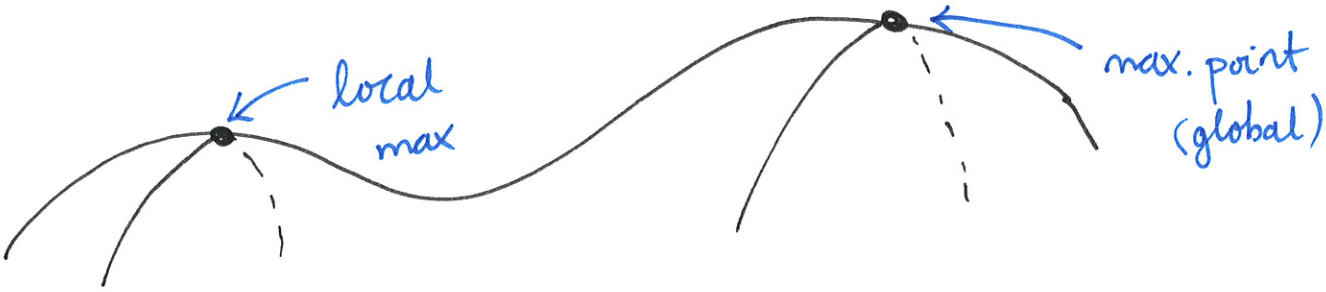
Optimization: max/min

Def (max/min):

i) (x^*,y^*) is a maximal point/maximizer for f if

$$f(x^*,y^*) \geq f(x,y) \text{ for all } (x,y) \in D_f.$$

ii) (x^*,y^*) is a local max. for f if $f(x^*,y^*) \geq f(x,y)$ for all (x,y) close to (x^*,y^*).

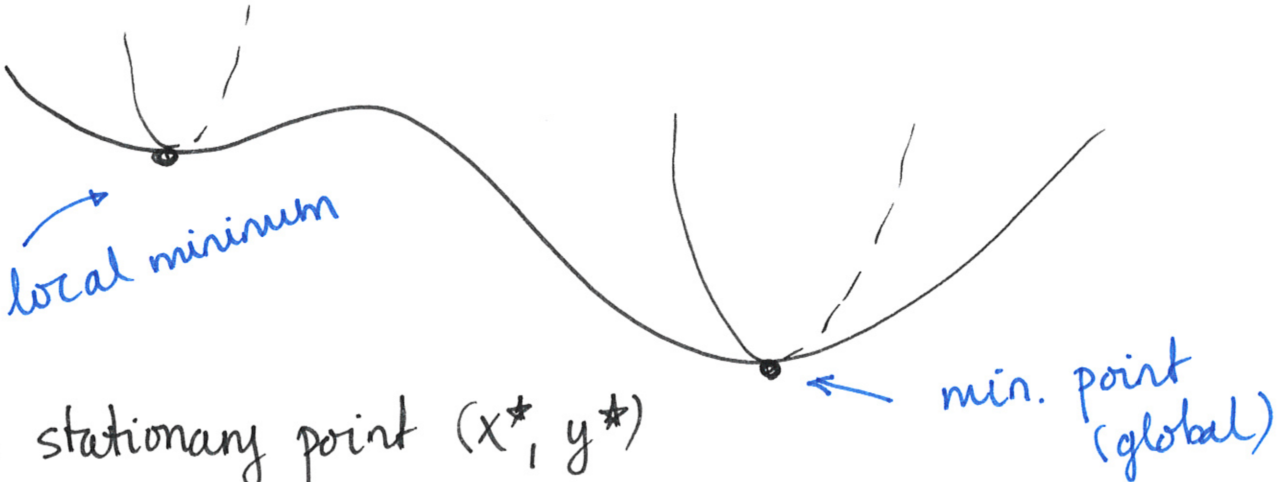


iii) (x^*, y^*) is a minimum point / minimizer for f

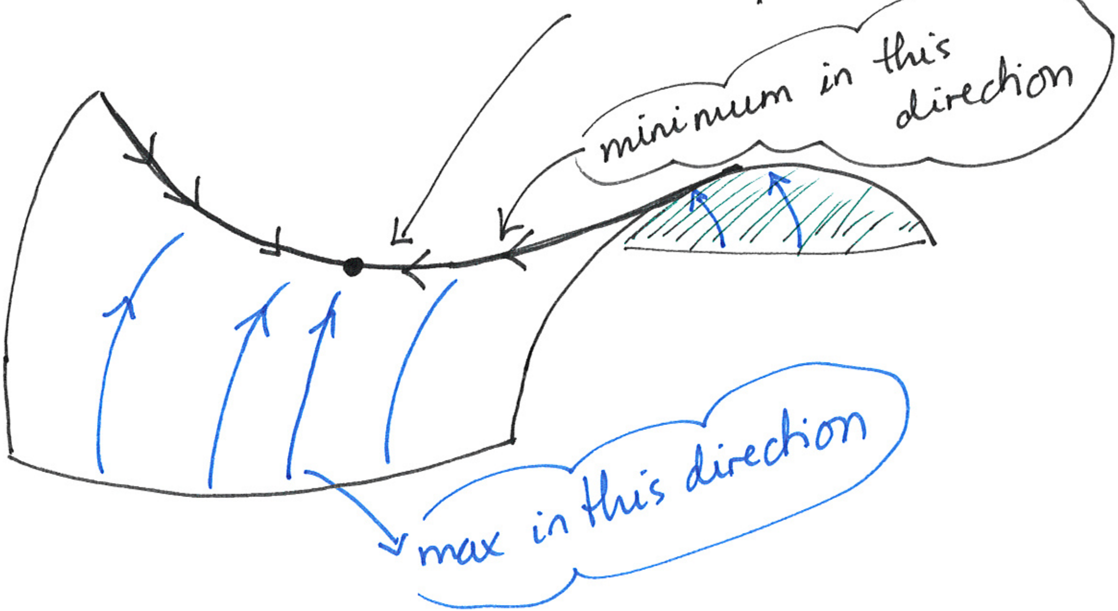
if $\quad f(x^*, y^*) \leq f(x, y)$ for all $(x, y) \in D_f$.

iv) (x^*, y^*) is a local minimum for f if

$f(x^*, y^*) \leq f(x, y)$ for all (x, y) close to (x^*, y^*).



v) A stationary point (x^*, y^*) of f which is neither a local max. nor a local min. is called a saddle point.



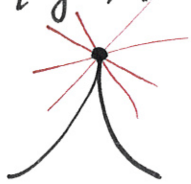
KEY RESULT: If (x^*, y^*) is a max/min. point for f, then we have either:

i) (x^*, y^*) is a stationary point for f

$$(f'_x = f'_y = 0 \text{ at } (x^*, y^*))$$

ii) Either f'_x or f'_y is not defined at (x^*, y^*).

iii) (x^*, y^*) is a boundary point of D_f.



The second derivative test

Result: If (x^*, y^*) is a stationary point for f, we get:

$$\mathcal{H}(f)(x^*, y^*) = \begin{bmatrix} f''_{xx}(x^*, y^*) & f''_{xy}(x^*, y^*) \\ f''_{yx}(x^*, y^*) & f''_{yy}(x^*, y^*) \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ \mathbf{B} & C \end{bmatrix}$$

$AC - B^2$

We have: 1) If $\det \mathcal{H}(f)(x^*, y^*) > 0$ and $\operatorname{tr} \mathcal{H}(f)(x^*, y^*) > 0$, then (x^*, y^*) is a local min.

"Trace": $A + C$

2) If $\det \mathcal{H}(f)(x^*, y^*) > 0$ and $\operatorname{tr} \mathcal{H}(f)(x^*, y^*) < 0$, then (x^*, y^*) is a local max.

3) If $\det \mathcal{H}(f)(x^*, y^*) < 0$, then (x^*, y^*) is a saddle point.

NB: If $\det \mathcal{H}(f)(x^*, y^*) = 0$, the test is inconclusive.