Inner product (dot product) of vectors

EBA 1180
Sect. 38
Spring 25

Def (Inner product): Let $\vec{v}, \vec{w}$ be n-vectors:

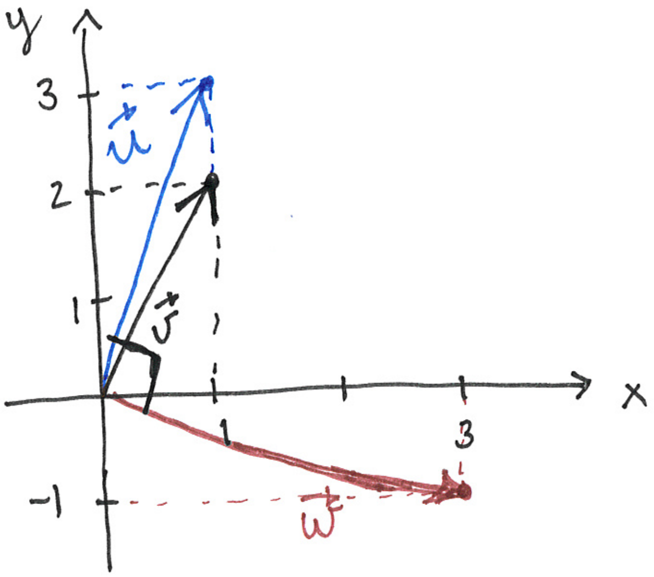
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}. \text{ Then,}$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\vec{v} \cdot \vec{w} = 1 \cdot 3 + 2 \cdot (-1) = 3 - 2 = \underline{1}$$

$$\vec{u} \cdot \vec{w} = 1 \cdot 3 + 3 \cdot (-1) = 3 - 3 = \underline{0}$$



NOTE: $\boxed{\vec{u} \cdot \vec{w} = 0}$

and $\vec{u} \perp \vec{w}$

"forms a 90° angle with"

"$\vec{u}$ and $\vec{w}$ are orthogonal"

Def (Orthogonal): We say that $\vec{v}$ and $\vec{w}$ are <u>orthogonal</u> if
$$\vec{v}\cdot\vec{w} = 0$$

<u>Interpret</u>: $\vec{v}$ and $\vec{w}$ form a $90°$ angle: $\vec{v}$ is <u>normal</u> to $\vec{w}$.

<u>Result</u>: $\vec{v} \perp \vec{w} \iff \vec{v}\cdot\vec{w} = 0$

Rules of computation:

1) $\vec{v}\cdot\vec{w} = $ a number

2) $\vec{v}\cdot\vec{v} = v_1^2 + v_2^2 + \dots + v_n^2 = \|\vec{v}\|^2 \geq 0$

$$\|v\|^2 = \left(\sqrt{v_1^2 + v_2^2 + \dots + v_n^2}\right)^2$$
$$= v_1^2 + v_2^2 + \dots + v_n^2$$

3) $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$

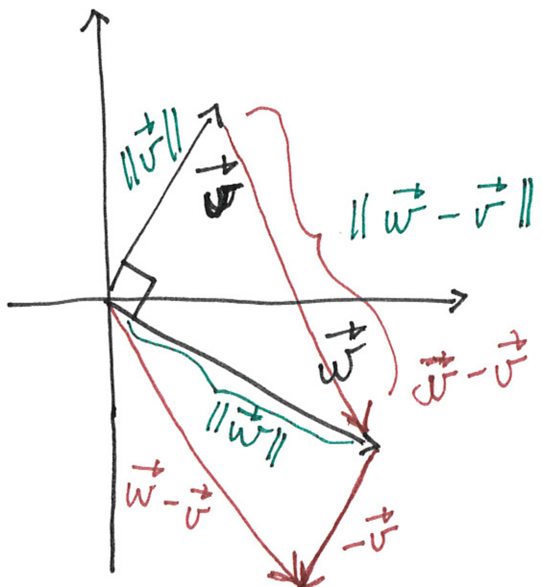
$(a\vec{u} + b\vec{w})\cdot\vec{v} = a\vec{u}\cdot\vec{v} + b\vec{w}\cdot\vec{v}$

Proof of result:

Pythagoras

$\vec{v}\perp\vec{w} \iff \|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{w}-\vec{v}\|^2$

$\Updownarrow$ Rule 2



$$(\cancel{v_1^2} + \dots + \cancel{v_n^2}) + (\cancel{w_1^2} + \dots + \cancel{w_n^2})$$
$$= (w_1 - v_1)^2 + \dots (w_n - v_n)^2$$
$$= \cancel{w_1^2} - 2w_1v_1 + \cancel{v_1^2} + \dots$$
$$+ \cancel{w_n^2} - 2w_nv_n + \cancel{v_n^2}$$

$$0 = -2w_1v_1 - \dots - 2w_nv_n$$

$$\Updownarrow$$

$$w_1v_1 + \dots + w_nv_n = 0$$

$$\vec{w} \cdot \vec{v} = 0$$

$$\Updownarrow \text{ Rule 3}$$

$$\vec{v} \cdot \vec{w} = 0 \qquad \blacksquare$$

<u>NOTE</u>: $\underbrace{\vec{v} \cdot \vec{w}}_{\text{inner product of n-vectors}} = \underbrace{\vec{v}^{\,T}\,\vec{w}}_{\text{matrix multiplication}}$

<u>EX</u>: $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$\vec{w} \cdot \vec{v} = 1 \cdot 2 + (-3) \cdot 1 = 2 - 3 = \underline{\underline{-1}}$$

SAME

$$\vec{w}^{\,T}\vec{v} = \begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 \cdot 2 + (-3) \cdot 1 = \underline{\underline{-1}}$$

$$1 \times \cancel{2} \cdot \cancel{2} \times 1$$

<u>NB</u>: $\underbrace{\vec{w}\,\vec{v}}_{\text{matrix multiplication}}$ is not defined: $\begin{bmatrix} 1 \\ -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$2 \times 1 \quad 2 \times 1$

Not same!

START: 10.58

Functions in two variables

<u>Ex</u>: $f(x, y) = 2x + 3y - 1$, linear function

two variables

$f(x,y) = x^2 + y^2$, polynomial function

$f(x,y) = \dfrac{x+y}{x-y}$, rational function

$f(x,y) = x e^{y}$

<u>General</u>: $f(x,y)$: function expression in x, y

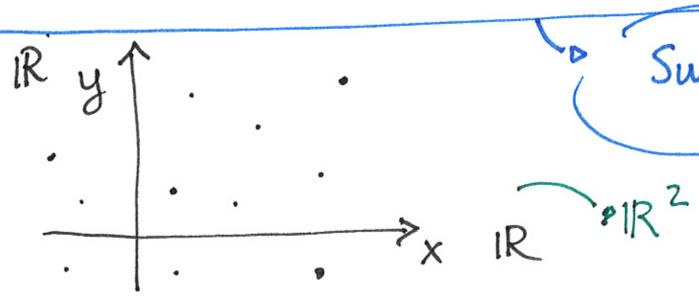
x, y : input variables

$z = f(x,y)$: output variable

<u>Def (Domain of f)</u>:

D_f = domain of f = all coordinate pairs (x,y) that we can input into function f

Subset of xy-plane, $\mathbb{R}^2$

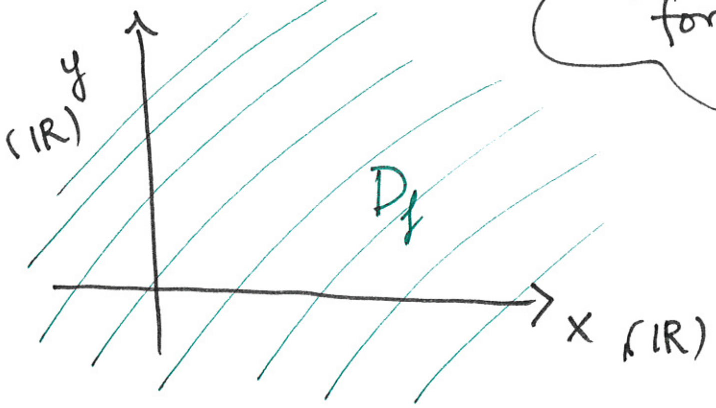


Ex: $f(x,y) = 2x + 3y - 1$

$D_f = \mathbb{R}^2$

"R two"

any real value for x, any real value for y

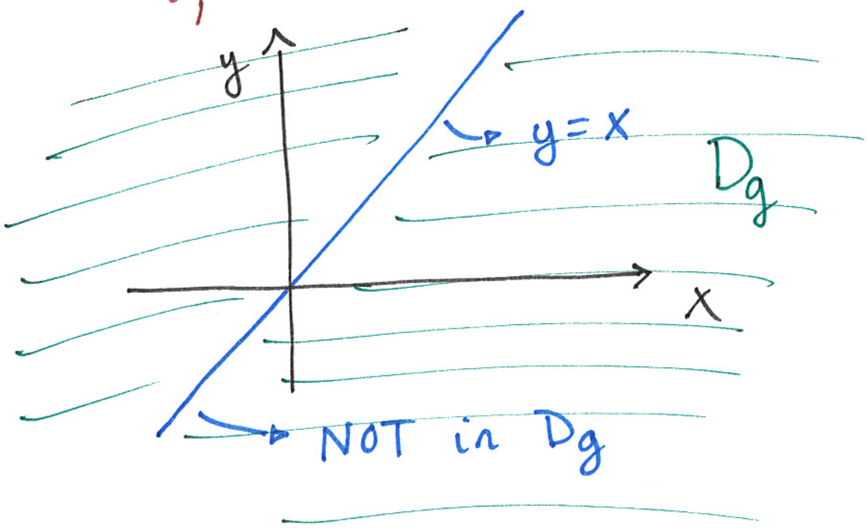


$g(x,y) = \dfrac{x+y}{x-y}$, D_g : $x \neq y$

$x \neq y$: Divide by 0 if $x = y$

$$\underbrace{D_g}_{\text{domain of } g} = \underbrace{\{}_{\text{the set of}} (x,y) \underbrace{\in}_{\text{element in}} \mathbb{R}^2 \underbrace{:}_{\text{such that}} x \neq y \}$$

(set of)



Natural domain

Def (Range): V_f = range of f = all values $f(x,y)$ can attain when $(x,y) \in D_f$

• To find the range: Find the max/min of f.

Ex: $f(x,y) = 2x + 3y - 1$

From ex: $D_f = \mathbb{R}^2$

$x \to \infty,\ y=0$: $f(x,y) \to \infty$

$x \to -\infty,\ y=0$: $f(x,y) \to -\infty$

$V_f = (-\infty, \infty) = \mathbb{R}$

$g(x,y) = x^2 + y^2$

$D_g = \mathbb{R}^2$

Can only get non-neg. numbers because of squares

$V_f = [0, \infty)$

Graphs and level curves

Def (Graph of function in two variables):
The graph of a function f in two variables is the set of all points

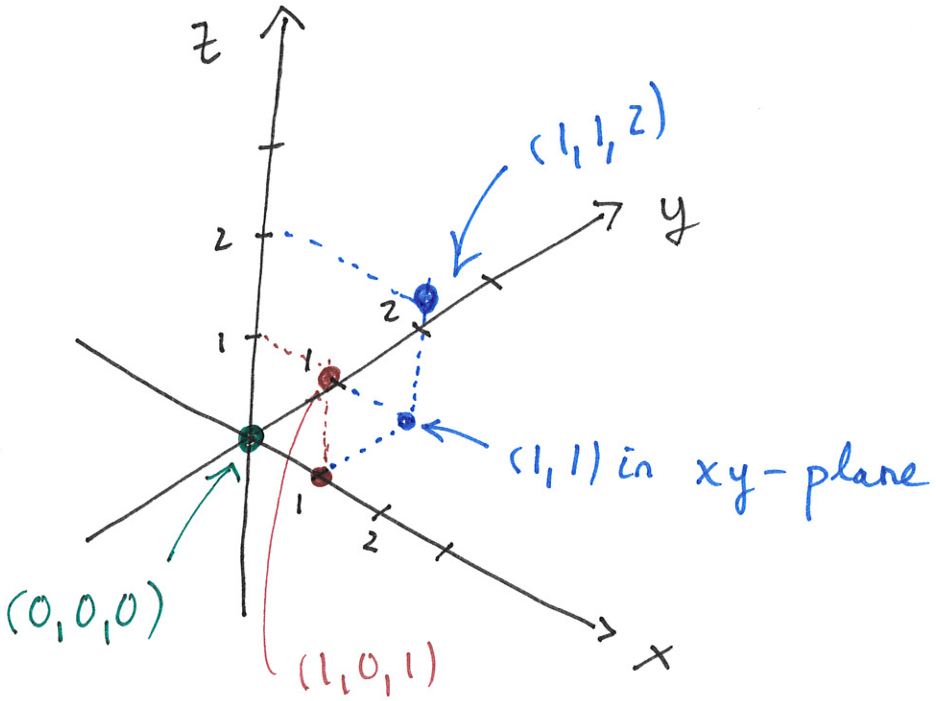
$$(x, y, z)$$

where $(x,y) \in D_f$ and $z = f(x,y)$.

• Can draw the graph of f in the xyz - coordinate system.

Ex: $f(x,y) = x^2 + y^2$, $D_f = \mathbb{R}^2$

(x,y)	$(0,0)$	$(1,0)$	$(1,1)$
$z = f(x,y)$	$0^2+0^2=0$	$1^2+0^2=1$	$1^2+1^2=2$
Point in xyz-plane	$(0,0,0)$	$(1,0,1)$	$(1,1,2)$



• The graph of f is called a <u>surface</u>.

Def (Level curves): All (x,y) such that $f(x,y) = c$ for a constant c.

<u>In general</u>: Graph of a function $f(x,y)$: