Exercise sheet 35

EBA 1180
Spring 23

Mandatory assignment:
March 13th
– March 20th
No lectures

3×2

5) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 \\ 1 & -2 \\ 7 & 1 \end{bmatrix}$

l) $C^T A = \underbrace{\begin{bmatrix} 3 & 1 & 7 \\ 4 & -2 & 1 \end{bmatrix}}_{2 \times \cancel{3}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{bmatrix}}_{\cancel{3} \times 3}$

$$= \begin{bmatrix} 3\cdot 1 + 1\cdot 2 - 7\cdot 1 & 3\cdot 1 + 1\cdot 1 + 7\cdot 1 & 3\cdot 1 + 1\cdot 4 + 7\cdot 1 \\ 4\cdot 1 + (-2)\cdot 2 + 1\cdot(-1) & 4\cdot 1 - 2\cdot 1 + 1\cdot 1 & 4\cdot 1 - 2\cdot 4 + 1\cdot 1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} -2 & 11 & 14 \\ -1 & 3 & -3 \end{bmatrix}}} \quad 2 \times 3$$

7) b), c), d):

Profit per stock

	A (x)	B (y)	C (z)
1	20	5	30
2	40	-50	180
3	-20	25	-265

Profit from owning 1 of stocks A, B, C respectively in scenario 3

Profit for owning 1 of stock A in each of the 3 scenarios.

- Budget constraint:

$$60x + 75y + 320z = C \quad 400\,000$$

Which profits are possible? Profits: (R_1, R_2, R_3)

Profit in scenario 1:

$$20x + 5y + 30z = R_1$$

Profit in scenario 2:

$$40x - 50y + 180z = R_2$$

Profit in scenario 3:

$$-20x + 25y - 265z = R_3$$

+ budget constraint: Gaussian elimination:

$$\begin{array}{c} \left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 40 & -50 & 180 & R_2 \\ -20 & 25 & -265 & R_3 \\ 60 & 75 & 320 & C \end{array}\right] \end{array} \begin{array}{l} \\ -2 \\ 1 \\ -3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 30 & -235 & R_3 + R_1 \\ 0 & 60 & 230 & C - 3R_1 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2} \\ 1 \end{array}$$

$$\sim \left[\begin{array}{ccc|l} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 0 & -175 & R_3 + \cancel{R_1} + \frac{1}{2}(R_2 - \cancel{2R_1}) \\ 0 & 0 & 350 & C - 3R_1 + (R_2 - 2R_1) \end{array}\right] \quad 2$$

$$\sim \left[\begin{array}{ccc|l} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 0 & -175 & R_3 + \frac{1}{2}R_2 \\ 0 & 0 & 0 & C - 5R_1 + R_2 + 2\left(R_3 + \frac{1}{2}R_2\right) \end{array}\right]$$

$$\| $$

$$C - 5R_1 + 2R_2 + 2R_3$$

For system to have a solution, must have:

$$C - 5R_1 + 2R_2 + 2R_3 = 0, \text{ i.e.,}$$

$$(*) \qquad 5R_1 - 2R_2 - 2R_3 = 400\,000$$

b) $\underline{(R_1, R_2, R_3) = (50', 25', -100')}$:

$\underline{(*)}$: $5 \cdot 50' - 2 \cdot 25' - 2 \cdot (-100') = 400'$; Holds! OK!

$$\begin{cases} 20x + 5y + 30z = 50' \\ \quad -60y + 120z = 25' - 2 \cdot 50' = -75' \\ \quad\quad -175z = 100' + \frac{1}{2}25' = -87{,}5' \end{cases}$$

$$\underline{\underline{z = 500}}$$

$$y = \frac{-75' - 120 \cdot 500}{-60} = \underline{\underline{2250}}$$

$$x = \frac{1}{20}(50' - 5y - 30z) = \underline{\underline{1187{,}5}}$$

c) $\underline{R_1 > 0,\ R_2 = R_3 = 0?}$

(★): $5R_1 = 400'$

$R_1 = 80'$, so

$(80', 0, 0)$ is possible. Corresponding portfolio:

Insert in echelon form of matrix

$$\begin{cases} 20x + 5y + 30z = 80' \\ \quad -60y + 120z = -160' \\ \quad\quad -175z = 0 \end{cases} \Rightarrow$$

$\underline{x = 3333{,}33\ldots}$, $\underline{y = 2666{,}66\ldots}$,

$$\underline{\underline{z = 0}}$$

d) $\underline{R_1, R_2, R_3 > 0?}$

(★): $5R_1 - 2R_2 - 2R_3 = 400'$

$\underline{\text{Try } R_1 = R_2 = R_3:}$ $5R_1 - 2R_1 - 2R_1 = 400'$

$R_1 = 400' > 0$

So $(R_1, R_2, R_2) = (400', 400', 400')$ is possible.

Other combinations also work, f. ex:

$$(R_1, R_2, R_3) = (100', 25', 25'):$$

$$\underline{(\star)}: \quad 5R_1 - 2R_2 - 2R_3 = 500' - 50' - 50'$$

$$= 400' \; ; \quad \text{OK!}$$

Inverse matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

$$\underline{|A|} = 1\,\overset{6}{(18-12)} - 1\,\overset{5}{(9-4)} + 1\,\overset{1}{(3-2)} = 2 \underline{\neq 0},$$

so A has an inverse. Use formula to find A^{-1}:

$$A^{-1} = \frac{1}{\underset{\det(A)}{|A|}} \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T}_{\text{Adjoint matrix of } A;\ \operatorname{adj}(A)}$$

Compute c_{ij}'s :

$c_{11} = 6 \qquad c_{12} = -5 \qquad c_{13} = 1$

$c_{21} = -6 \qquad c_{22} = 8 \qquad c_{23} = -2$

$c_{31} = +2 \qquad c_{32} = -3 \qquad c_{33} = +1$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}^T$$

$$= \frac{1}{2}\begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Alternative way of finding A^{-1}

A $n \times n$ matrix

$$[A \mid I] \sim \ldots \sim [B \mid C]$$

Reduced echelon form (aim for $B = I$)

2 cases:

$B = I$: $A^{-1} = C$

$B \neq I$: A^{-1} doesn't exist

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $|A| = 4 - 1 = 3 \neq 0$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Seen before!

Alternative method:

$$[A \mid I] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \text{ switch}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \xleftarrow{-2} \sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{array}\right]$$

Echelon form

Divide row 2 by -3

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \xleftarrow{-2}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

Reduced echelon form!

$$= [I \mid A^{-1}], \text{ so}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Why does this work?

3×3

Some write $\rightarrow$ for row red.

$$A = \begin{bmatrix} 1 & \cdots & \\ 2 & \cdots & \\ \cdots & \cdots & \end{bmatrix} \xleftarrow{-2} \sim \begin{bmatrix} 1 & \cdots & \\ 0 & \cdots & \\ \cdots & \cdots & \end{bmatrix} = B, \text{ then}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow -2 \\ \\ \end{matrix}$$

$$B = E_1 \cdot A$$

elementary matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \text{ the matrix}$$

E_1 is found by doing the elementary row operation to the identity matrix.

$$[A \mid I] \sim \quad \ldots \quad \sim [B \mid C]$$

$$\Updownarrow$$

$$\left.\begin{aligned} &E_r \ldots E_2 E_1 A = B \\ \text{and} \quad &E_r \ldots E_2 E_1 I = C \end{aligned}\right\} \Rightarrow$$

Multiply with <u>one</u> elementary matrix for each row op.

<u>Then, if $B = I$:</u>

$$\begin{cases} E_r \ldots E_2 E_1 A = I \\ E_r \ldots E_2 E_1 = C \end{cases}$$

<u>If $B \neq I$:</u> $\quad |A| = 0 \Rightarrow$ No inverse.

Inner product (dot product) of vectors

Def (Inner product): Let $\vec{v}, \vec{w}$ be n-vectors

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}. \text{ Then,}$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\vec{v} \cdot \vec{w} = 1 \cdot 3 + 2 \cdot (-1) = 3 - 2 = \underline{1}$$

Read: "$\vec{v}$ DOT $\vec{w}$"